GEOMETRICAL DRAWING.

PART II.

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GEOMETRICAL DRAWING.

PART II.

THE PRACTICAL GEOMETRY OF PLANES AND SOLIDS:

COMPRISING THE

ELEMENTS OF DESCRIPTIVE GEOMETRY, WITH
ITS APPLICATION TO HORIZONTAL AND ISOMETRIC PROJECTION,
AND THE PROJECTION OF SOLIDS AND SHADOWS.

With nearly 300 Exercises.

DESIGNED FOR THE

USE OF STUDENTS PREPARING FOR EXAMINATION.

BY

SAMUEL H. WINTER.

NEW EDITION, REVISED AND ENLARGED.

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PREFACE

TO

THE PRESENT EDITION.

THIS, the Third Edition, has been carefully corrected, and enlarged by the insertion of Papers set at the Examinations for admission to the Royal Indian Civil Engineering College, Cooper's Hill, during the years 1876, 1877, 1878, 1879, &c.

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January 1880.

CONTENTS.

CHAPTER I.

CHAPTER II.

ELEMENTARY PROBLEMS ON STRAIGHT LINES AND PLANES.

CHAPTER III.

HORIZONTAL PROJECTION.

CHAPTER IV.

THE PROJECTION OF SOLIDS.

CHAPTER V.

ON SHADOWS.

CHAPTER VI.

CHAPTER VII.

PRACTICAL GEOMETRY OF PLANES AND SOLIDS.

CHAPTER I.

INTRODUCTION.

1. The object of DESCRIPTIVE GEOMETRY is to resolve, by means of constructions in a single plane, problems involving three dimensions of magnitude. This end is effected by the *Method of Projections*, which consists in referring points in space to two given planes intersecting at a known angle.

2. The given planes are called the PLANES OF PROJECTION.

3. The PROJECTION OF A POINT is the point in which the perpendicular, drawn from the point to the plane of projection, meets the plane.

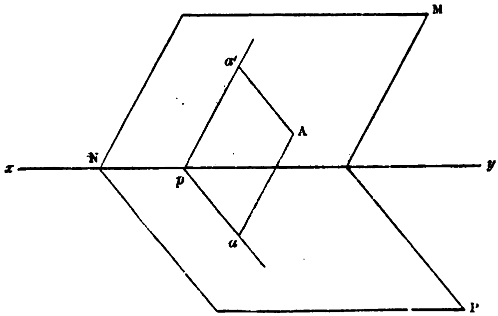
4. This perpendicular is called the PROJECTOR of the point.

Thus if A (Fig. 1) be a point in space, M N and N P the planes of projection, cutting each other in $x\,y$, and A a be drawn perpendicular to the plane N P, a will be the projection of A on N P; A a, its projector. Similarly, A a' and a' are its projector and projection, with reference to the plane M N.

5. When the projectors are perpendicular to the planes of projection the projection is said to be ORTHOGRAPHIC.

6. The PROJECTION OF A LINE is the line which passes through the projections of all points in that line.

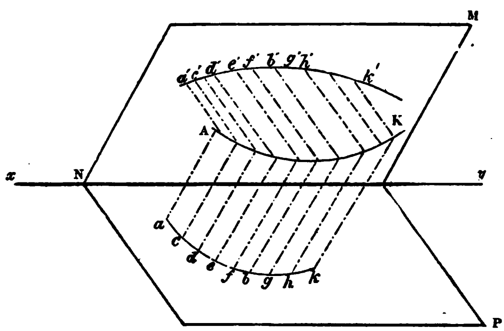
Fig. 1.



7. The PROJECTING SURFACE OF A LINE is the surface, plane or otherwise, which contains the projectors of all points in that line.

If A K (Fig. 2) be any line in space *a*, *c*, *d*, *e*, *f*, *b*, &c., the projections of its several points on the plane N P, the line

Fig. 2.



a c d e f b g h k, which passes through all these points, will be the projection of A K on the plane N P, and the surface A *a k* K will be its projecting surface.

The projection of a line always contains the projections of all points and lines on its projecting surface.

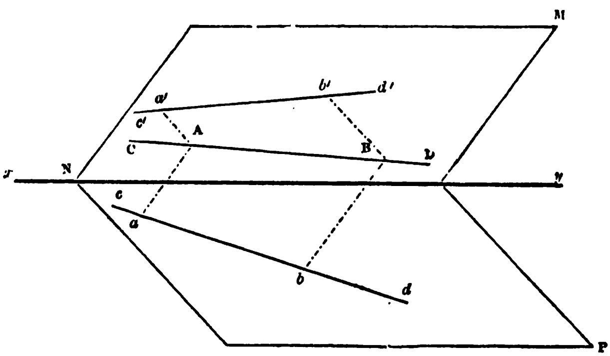
8. THEOREM.—*The projection of a straight line is a straight line.*

For the *projectors* of all its points, being drawn from points in a straight line, and being also perpendicular to the plane of projection, are in one plane (*Euc.* XI. 6, and I. *def.* 35). The intersection of this plane with the plane of projection being the line which passes through the projections of all points in the given line, is the projection of that line (6), and it is a straight line (*Euc.* XI. 3).

It is thus evident that the projecting surface of a straight line is a PLANE. Also that the projection of a straight line is determined by the projections of any two points in that line, since only one straight line can pass through the same two points.

In Fig 3, let *a* and *b* be the projections of A and B, any two points, in the indefinite straight line C D, in space; the straight

Fig. 3.



line *c d*, passing through *a* and *b*, will be the projection of C D, on the plane N P.

9. Since A *a* and B *b* are perpendicular to N P, the plane A *a b* B is perpendicular to N P (*Euc.* XI. 18). Consequently the projecting surface of a straight line is the plane which contains the line, and is perpendicular to the plane of projection.

10. If a straight line is perpendicular to the plane of projec-

tion, its projection is a point, since the projectors of all points in the line coincide with the line itself.

11. If two straight lines are parallel, their projections upon the same plane are parallel (*Euc.* XI. 16). In order, therefore, to construct the projections of two parallel lines, it will be sufficient to know the projections of two points in one of them, and of one point in the other.

12. When the line projected is not a straight line, its projecting surface will generally be what is termed a cylindrical one, and its projection a curve. If, however, the curve is a plane curve, and its plane is perpendicular to the plane of projection, its projection will manifestly be a straight line.

13. If a line is situated in a plane parallel to the plane of projection, its projection is equal to the line itself. In the case of a straight line this is evident from *Euc.* I. 33. In the case of a curve, the line and its projection may be considered as the intersections of a cylinder by two parallel planes.

14. THEOREM.—*If two planes cut each other, and a point be projected on both of them, the perpendiculars drawn from the projections to the intersection of the planes will meet that intersection in the same point.*

Let the planes M N and N P (Fig 1) cut each other in $x\ y$.

Let $a\ a'$ be the projections of a point A in space, upon N P and M N respectively.

Then the plane a' A a will be perpendicular to M N and N P (*Euc.* XI. 18), and will cut $x\ y$ in some point p; consequently (*Euc.* XI. 19), $x\ y$ is perpendicular to the plane a A a'; the angles $y\ p\ a$ and $y\ p\ a'$ are therefore right angles (*Euc.* XI. *def.* 3). Now the straight lines $p\ a$ and $p\ a'$ pass through a and a', the projections of A; therefore the perpendiculars drawn from a and a' must meet $x\ y$ in the same point.

Conversely. When two points, one in each plane of projection, are so situated that the perpendiculars drawn from them to the intersection of those planes meet the intersection in the same

point, these two points may be considered as the projections of a single point in space.

15. A straight line is generally determined by its projections on two given planes, since the line itself is the intersection of its projecting planes. When however the projections are perpendicular to the intersection of the planes, a third plane of projection will be necessary to determine the line, since the projections may be those of any line in a plane perpendicular to both planes of projection.

16. Two straight lines assumed arbitrarily, one in each plane of projection, can be considered as the projections of the same straight line in space, only when the planes containing those lines, and perpendicular respectively to the planes of projection, are not parallel. Similarly, two curves, one in each plane of projection, can be considered as the projections of the same curve in space, only when the cylindrical surfaces passing through these curves, and perpendicular respectively to the planes of projection, cut each other. The curve itself will be the intersection of these surfaces.

17. THE TRACE OF A LINE is the point in which it meets the plane of projection.

18. THE TRACE OF A PLANE is the line in which it intersects the plane of projection.

19. A plane is determined by its traces, since only one plane can pass through the same two straight lines.

20. The intersection of the planes of projection is called the GROUND LINE OR AXIS.

21. If a plane is not parallel to the ground line it must meet it in a point common to both of its traces.

22. If a plane is parallel to the ground line its traces are also parallel to the ground line.

This is evident, for since the axis is parallel to the plane it

cannot meet it, and consequently cannot meet the traces which are lines in the plane; but each trace and the axis are in one plane, therefore they are parallel (*Euc.* I. *def.* 35).

If a plane is perpendicular to the axis, its traces are perpendicular to the axis (*Euc.* XI. 19, and *def.* 3).

If a plane is parallel to one plane of projection its trace upon the other is parallel to the ground line (*Euc.* XI. 16).

If a plane contains the axis, a third plane of projection is necessary to determine it.

23. No reference has yet been made to the magnitude of the angle contained by the planes of projection. In what follows this angle will be assumed to be a right angle, since that supposition simplifies the constructions employed in the solution of problems.

For the sake of distinction, one plane of projection will be called the VERTICAL PLANE, the other the HORIZONTAL.

24. Traces and projections are called VERTICAL or HORIZONTAL accordingly as they are in the vertical or the horizontal plane. A vertical projection is sometimes called an ELEVATION, and a horizontal projection is called a PLAN.

25. A HORIZONTAL LINE is a line parallel to the horizontal plane.

26. A VERTICAL LINE is a line perpendicular to the horizontal plane.

27. The following consequences result from assuming the planes of projection at right angles to each other.

I. *The projections of all points in the planes of projection are in the ground line.*

II. *The projections of all lines situated in a plane parallel to one of the planes of projection are parallel to the ground line.*

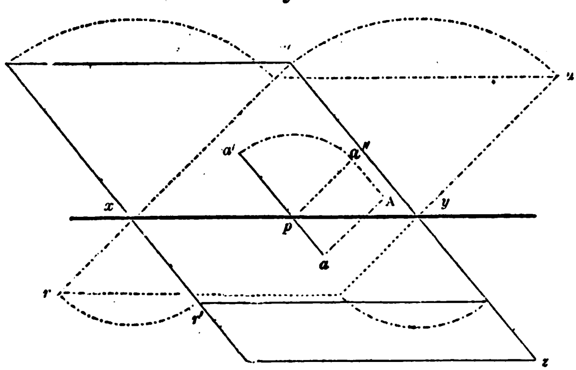
III. *If a plane be perpendicular to one plane of projection, and not parallel to the other, its trace upon that other is perpendicular to the ground line* (*Euc.* XI. 19, and *def.* 3).

IV. *The distance of the elevation of a point from the ground line shows the distance of the point from the horizontal plane: the distance of the plan of a point from the ground line shows the distance of the point from the vertical plane.*

28. From the foregoing explanations it might be inferred that two planes are necessary in order to represent plans and elevations in their real magnitude. Such, however, is not the case, since all constructions required may be united in one drawing and on a single plane, by supposing one of the planes of projection to revolve about the ground line until it coincides with the other plane of projection. In practice it is usual to turn the vertical plane back about the ground line, until it coincides with the horizontal plane.

Thus, in Fig. 4, the vertical plane, $x\,y\,u$, revolves about $x\,y$ until it takes the position $x\,y\,u'$, when $x\,y\,z$, the horizontal plane,

Fig. 4.



and $x\,y\,u'$ form one plane. All constructions will now be made in the horizontal plane, and in order to arrive at a correct idea of the relative positions of points, given by their projections only, the vertical plane must be conceived to take its original position. For example, if a a' be the projections of a point A in space, the point a' coming to the position a''; and the plane $x\,y\,u'$ be supposed to make one-fourth of a revolution about $x\,y$,

so as to take the position $x\ y\ u$, and from a and a'' two straight lines be drawn perpendicular to $x\ y\ z$ and $x\ y\ u$ respectively, these perpendiculars will intersect in a point which will be A.

29. After the vertical plane has been turned down to coincide with the horizontal plane, that portion of it which was below the ground line, as $x\ y\ r$, will take the position $x\ y\ r'$. Any elevations on it will therefore be in front of the ground line.

It is evident that points in space will be represented by their projections only.

30. THEOREM.—*The straight line which joins the plan and the elevation of a point is perpendicular to the axis.*

This follows at once from (14); for during the revolution of $x\ y\ u$ about $x\ y$ (Fig. 4), $p\ a''$ remains perpendicular to $x\ y$; therefore (*Euc.* I. 14), $a\ a'$ is a straight line, and it is perpendicular to $x\ y$.

31. THEOREM.—*If a straight line is perpendicular to a plane its projections are respectively perpendicular to the traces of that plane.*

For, the trace of the given plane is the line in which it cuts the plane of projection (18);

The projecting plane of the line is perpendicular to the plane of projection (9), and also to the given plane (*Euc.* XI. 18).

The trace is therefore perpendicular to the projecting plane (*Euc.* XI. 19); and

Consequently, to the projection of the line (*Euc.* XI. *def.* 3).

32. A DIHEDRAL ANGLE is the angle contained by two intersecting planes.

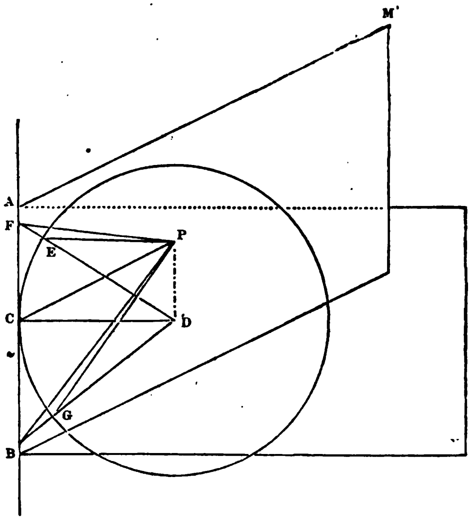
33. THE PROFILE ANGLE of two planes is the angle contained by the two straight lines in which these planes are cut by a third plane, at right angles to both of them. This third plane is called a PROFILE PLANE. Since these lines are perpendicular to the intersection of the two given planes (*Euc.* XI. 19, and *def.* 3),

the profile angle will be the measure of the dihedral angle (*Euc.* XI. *def.* 6).

34. THEOREM.—*The profile angle of two planes is the greatest angle which any straight line in one plane can make with the other plane.*

Let A B (Fig. 5) be the intersection of two planes, M A B, N B A, P C D their profile angle (33). In A B take any point

Fig. 5.



F, draw P D, perpendicular to C D, it will be perpendicular to the plane N B A (*Euc.* XI. *def.* 4), join P F and D F, make D E equal to D C, and join P E. Then (*Euc.* I. 4) the angle P C D is equal to the angle P E D; but the angle P E D is greater than the angle P F D (*Euc.* I. 16), therefore the angle P C D is greater than the angle P F D, which is the inclination of a line P F in the plane M A B to the plane N B A; and the same may be proved of any other such line.

Cor. 1. Let E C G be a circle, whose centre is D and radius D C, then, since the angle P C D is equal to the angle P E D equal to the angle P G D, it is evident that all straight lines

passing through a given point, and making a given angle with a given plane, meet that plane in the circumference of a circle whose centre is the point in which the perpendicular drawn from the point to the plane meets the plane.

Cor. 2. The projection of the line P C on the plane N B A is the line C D, which, by plain trigonometry, is equal to the line P C multiplied by the cosine of the angle P C D. If, therefore, L be the length of a straight line, θ the angle at which it is inclined to the plane of projection, P the projection,

$$P = L \times \cos\theta.$$

35. The *Notation* employed in Descriptive Geometry should be simple and uniform. In the following problems points in space will be denoted by italic capitals, as *A*, *B*, *C*; the plans of these points by the corresponding small letters, as *a*, *b*, *c*; their elevations by the same small letters distinguished by an accent, thus, a', b', c'.

The point (a, a') will therefore denote the point *A* in space, whose plan is *a*, and elevation a'.

The line $(a\,b, a'\,b')$ will denote the line *A B* in space, whose plan and elevation are $a\,b$ and $a'\,b'$ respectively.

36. It has been shown (28) that the constructions in Descriptive Geometry are all brought into one plane by supposing the vertical plane of projection to revolve about the ground line until it coincides with the horizontal plane. A similar process, viz., turning any plane about one of its traces until it coincides with the plane containing that trace, often facilitates the solution of problems.

This method of solving problems (called by French writers *La méthode des Rabattements*) depends upon the following principle:—

After a plane has been turned about one of its traces until it coincides with the plane of projection containing that trace, any point in the plane will be situated in a straight line drawn through the projection of the point, perpendicular to the trace. The distance of such point from the trace will be equal to the hypothenuse of a right-angled triangle, whose base is the distance of the pro-

jection from the trace, and whose perpendicular is the distance of the point from the plane of projection containing the trace.

This will be made evident by drawing a profile plane through the point, for then the projector of the point will be the perpendicular of the triangle, the intersection of the profile plane with the plane of projection will be its base, and the intersection of the profile plane with the given plane will be its hypothenuse. After the revolution of the plane this hypothenuse will either coincide with the base, or be in the same straight line with it, and the point in question will be at the extremity of the hypothenuse in its new position.

The application of this method to a given plane will now be explained.

I. *When the given plane is perpendicular to both of the planes of projection.*

II. *When it is perpendicular to one of them only.*

III. *When it is perpendicular to neither of them.*

I. Let (a, a') (Fig. 6) be the point situated in the plane M N P, which is perpendicular to both of the planes of projection. The traces M N and N P will form a straight line perpendicular to $x\,y$.

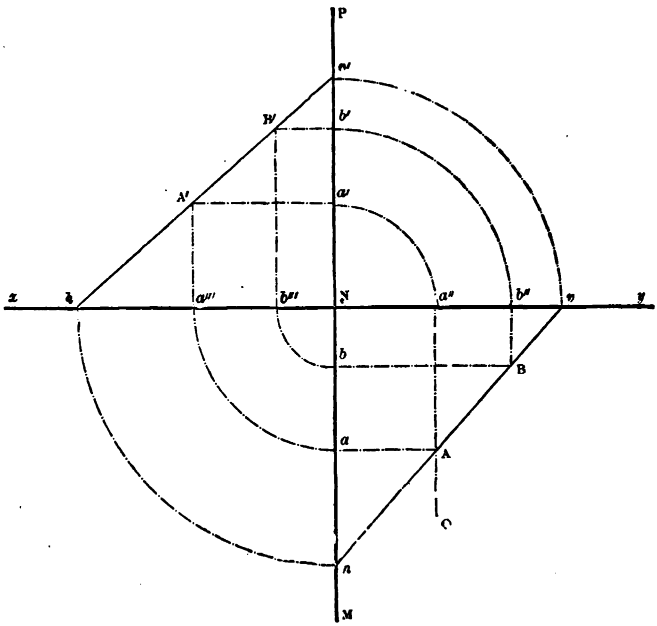
In this case, when the plane has been turned about its horizontal trace M N, the distance of the point A from that trace will evidently be equal to N a'. If, therefore, with N as a centre, and a radius N a', a circle a' a'' be described, cutting $x\,y$ in a'', and a'' O be drawn perpendicular to $x\,y$, and a A perpendicular to M N, A will be the point required. In the same manner a second point B (b, b') may be found, and thus the straight line A B containing these points is determined.

By a similar construction A′ and B′ may be found when the plane has been turned about its vertical trace, as shown in the figure.

Conversely. If A be given to find a and a', draw A a perpendicular to M N, and cutting it in a; draw A a'' perpendicular to $x\,y$, and meeting it in a''; with N as a centre, and a radius N a'',

describe the circle $a'' \, a'$, cutting N P in a'; a and a' will be the projections of A.

Fig. 6.



II. Let the point (a, a') (Fig 7) be situated in the plane M N P, which is perpendicular to the horizontal plane only. N P will be perpendicular to $x \, y$ (*Euc.* XI. 19).

If the plane be turned about its horizontal trace M N, the distance of the point A from M N will evidently be equal to $a \, a'$, $a \, a'$ being a straight line perpendicular to $x \, y$ (30).

Draw a A perpendicular to M N, make it equal to $a \, a'$; A will be the position of the point (a, a') after the plane has been turned down. In the same manner B (b, b') may be found; and thus the straight line A B passing through these points is determined.

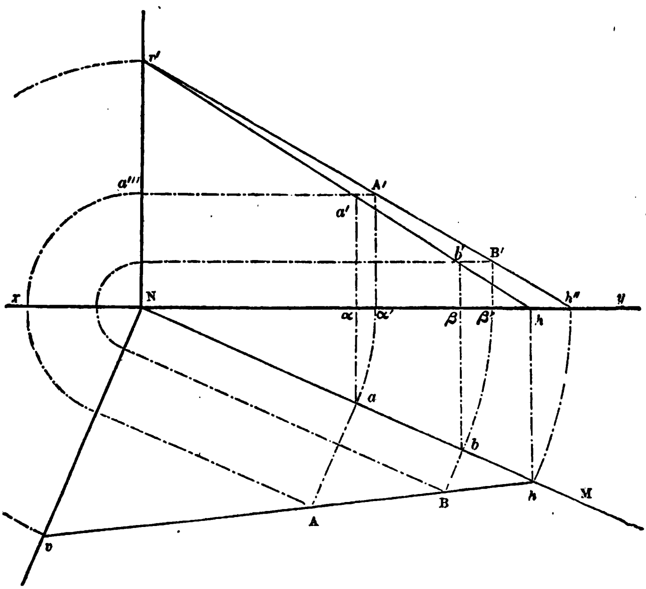
Conversely. If A be given to find a and a'.

Draw A a perpendicular to M N; a will be the plan of A; its

elevation will be in $a\ a'$, drawn from a perpendicular to $x\ y$ (30), and at a distance $\alpha\ a'$ from $x\ y$, equal to A a.

Again; if the plane be turned about its vertical trace. The point ($a\ a'$) will be situated in a straight line drawn through a' perpendicular to N P; the distance of A′ from N P will, by the principle enunciated (36), be equal to N a, the hypothenuse of

Fig. 7.



the right-angled triangle N $a\ \alpha$; because, in this case, the profile plane is parallel to the horizontal plane. In the same way B′ may be found, and the straight line A′ B′ determined. The point v', in which A′ B′ meets N P, will be the vertical trace of the line ($a\ b$, $a'\ b'$): the point h'', in which A′ B′ meets $x\ y$, will be the position which the horizontal trace of the line takes after the plane has been turned into coincidence with the vertical plane. If the constructions have been correctly performed the circle described with N as a centre, and radius N h, will pass through

h''; h being the point in which h' h drawn from h' perpendicular to x y cuts M N; h' being the point in which a' b' produced meets x y.

Conversely. To determine a and a' when A′ is given. Draw A′ a''' perpendicular to N P; a' will be in this line; draw A′ a' perpendicular to x y; with centre N, and radius N a', describe a circle cutting M N in a, a will be the plan of A; through a draw a a' perpendicular to x y, and cutting A′ a''' in a'; a' will be the elevation of A.

III. Let the point (a, a') (Pl. I. Fig. 1) be in the plane M N P, which makes oblique angles with both of the planes of projection, and is not parallel to the ground line.

Referring to the principle set forth (36), the point (a, a') will, after the plane has been turned about its horizontal trace, be situated in the straight line drawn from a perpendicular to M N, its distance from M N being equal to the hypothenuse of a right-angled triangle, whose base is a a'', and perpendicular a a'. This triangle may be constructed on the horizontal plane, as a a'' n: or on the vertical plane, as a a' m. With centre a'', and radius a'' n, describe a circle cutting a a'' produced in A. A will be the point required. A second point B may be found in the same manner, and thus the straight line A B determined.

Cor. It is evident that each of the angles a' m a and a a'' n measures the inclination of the plane M N P to the horizontal plane.

As a particular case of this problem let it be required to construct the vertical trace N P. Find, as above, the position V of (v, v') a point in the vertical trace; join N V; N V will be the line required, and the angle M N V will be the angle contained by the traces in space.

Conversely. Let a and a' be required when A is given. The plane may be given either by its traces M N and N P; or by its vertical trace N P, in the position N P′, and its horizontal trace M N. In both cases the right-angled triangle t v r, corresponding to the construction of V (v, v'), may be described. For, if M N and N P be given, v r=v v'; and t v are known. If M N and N P′ be given, t v and t r=t V are known.

Draw A a'' perpendicular to M N; and construct the right-

angled triangle $a''a\,n$, in which $a''\ n = a''$A; and the angle $a\,a''\,n =$ the angle $v\,t\,r$: a will be the plan of A; its elevation will be in $a\,a'$, perpendicular to $x\,y$, at a distance from $x\,y$, $a\,a' = a\,n$.

37. Every surface may be considered as having been generated by the movement of some line, constant or variable in magnitude. Whatever be the surface under consideration, the elements necessary to determine it must always be found in its definition, in its generation, or in its properties. So that, these elements being known, the solutions of all problems relative to that surface may readily be deduced therefrom.

38. The generating line is called the *generatrix* of the surface. The movement of the generatrix is regulated by one or more fixed lines; these lines are called the *directrices* of the surface. A surface may be represented graphically by the lines which constitute its generation; a mode of representation remarkable alike for elegance and simplicity, as will hereafter be seen. Since, however, each particular surface is capable of being generated in various ways, that method of representation should be chosen which is simplest, and best adapted to the problem under discussion.

A plane may be regarded as if generated by the movement of a straight line resting on both of the traces; or by a straight line which, resting on one trace, or on any straight line in the plane, moves parallel to the other trace. In this case, should the generating line move parallel to the horizontal trace, the plane would be represented by a series of straight lines parallel to the horizontal trace. A plane may also be generated by a straight line turning perpendicularly about a fixed straight line which it always meets in the same point.

39. *A spherical surface* is generated by the revolution of a circle about one of its diameters which remains fixed. Here the generatrix is a circle, constant in magnitude; the fixed diameter is the directrix, and the surface may be represented as shown in Fig. 7*a*. The spherical surface may also be considered as gene-

rated by a circle, whose plane is perpendicular to a diameter of the sphere, moving so that its centre is always in that diameter; its radii, in its various positions, being the corresponding semi-chords of the great circle of the sphere. In this case the directrix is the diameter containing the centres, whilst the generatrix is a circle variable in magnitude. The surface may be represented as shown in Fig. 7*b*.

Fig. 7*a*.

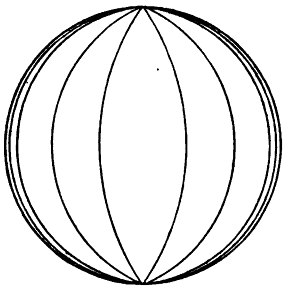
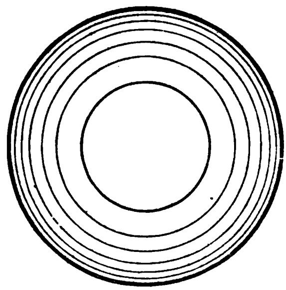


Fig. 7*b*.



40. *A cylindrical surface* is generated by a straight line which, resting upon some curve, moves parallel to itself. From this method of generation it follows that the intersection, or intersections, of a cylindrical surface by a plane parallel to the generatrix, will be one or more straight lines. A cylindrical surface may also be generated by a plane curve, which, moving parallel to itself, has one of its points resting on a fixed straight line. If a cylindrical surface be cut by two planes perpendicular to the generating straight line, the portion intercepted between the planes is termed a *right cylinder*; and, if the directrix be a circle, the cylinder will be the same as that generated by the revolution of a rectangle about one of its sides.

41. *A conical surface* is generated by a straight line, which, passing through a fixed point called the *vertex*, moves about that point in accordance with some law. It is, therefore, evident that the intersections of a conical surface by a plane passing through the vertex, will be two straight lines. When the generatrix moves about the vertex in such a manner that a second point in

it describes a circle, whose plane is perpendicular to the straight line passing through its centre and the vertex, the cone is called a right cone. Such a cone would be represented by a series of concentric circles, the distance between two consecutive ones being constant. The *axis* of a cone is the straight line drawn through the vertex and the centre of the directrix. The *sheets* of a cone are the surfaces separated by the vertex.

42. A curve surface may be considered as a polyhedron, whose faces are infinitely small; a *tangent plane* will then be nothing more than one of these faces indefinitely extended. Thus, to obtain a clear conception of a tangent plane to a surface in a point, imagine an infinitely small portion of the surface around the point to be taken; this may be considered as lying in one plane, and that plane indefinitely extended will be the tangent plane. Or thus: conceive any number of lines drawn through the point of contact on the surface, and take infinitely small portions of such lines around the point; these will all be situated in the tangent plane. These small portions of the lines may be considered as straight lines, which, if produced indefinitely, will be tangents to lines on the surface. An infinitely small portion of each of these tangents lies in the tangent plane; they must therefore lie wholly in that plane. The tangent plane to a surface in a point is therefore the plane containing the tangents, drawn through the point of contact, to all lines that can be drawn on the surface through the point of contact.

43. It is manifest that, if a plane touch a cylinder or a cone in any point, the generating line passing through that point lies entirely in the tangent plane: this property will be made available hereafter in drawing tangent planes. It is further evident that, if a cone be a right cone with a circular base, the tangent plane will be inclined to the base at the same angle as the generatrix, since the tangent plane will be perpendicular to the plane passing through the point of contact and the axis.

Examination on Chapter I.

1. Define the terms—'trace of a line,' 'trace of a plane,' 'projecting line,' 'projecting plane,' 'plane of projection,' 'section.'

2. Prove that if two straight lines are parallel, their plans are parallel.

3. In what case is the plan of an angle equal to the angle? Show when it is greater than the angle and when less.

4. The vertical trace of a plane makes an angle of 40° with $x\,y$; the angle between the traces is 60°: determine the horizontal trace of the plane.

5. By what data can the position in space of a point, of a straight line, of a plane, be determined?

6. If a straight line be parallel to a plane, every plane containing the line will either be parallel to the plane or cut it in a line parallel to the line. Prove this.

7. Show that the sections of two opposite faces of a cube by a plane are parallel.

8. Show that, if a right prism, or cylinder, be cut by a plane perpendicular to its base, the section is a rectangle.

9. The section of a prism, or a pyramid, by a plane parallel to its base, is a figure similar to the base. Prove this.

10. Prove that the perpendicular drawn from a point to a plane is the shortest distance from the point to the plane.

11. Show that if a straight line, not in a plane, be parallel to a straight line in that plane, it will also be parallel to the plane.

12. If a straight line be perpendicular to a plane, every straight line perpendicular to that line will be parallel to the plane.

CHAPTER II.

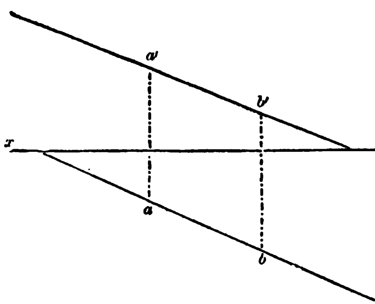
ELEMENTARY PROBLEMS ON STRAIGHT LINES AND PLANES.

PROBLEM I.

GIVEN the projections of two points, to find the projections of the straight line passing through the points.

Let (a, a') and (b, b') (Fig 8) be the points.

Fig. 8.



Then, since the projections of the straight line A B must pass through the projections of the points, the line $a'\ b'$ will be the elevation, and $a\ b$ the plan of the line required.

PROBLEM II.

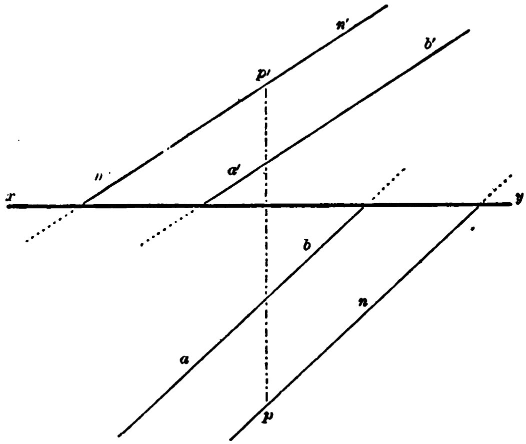
Through a given point, to draw a straight line parallel to a given straight line.

Let (p, p') (Fig. 9) be the given point, $(a\ b,\ a'\ b')$ the given line.

Since the required line passes through the point (p, p'), its

plan and elevation will pass through p and p' respectively; moreover the projections of parallel straight lines are parallel (11).

Fig. 9.



The elevation of the line will therefore be m' n' drawn through p' parallel to a' b'; its plan will be m n drawn through a parallel to a b.

Problem III.

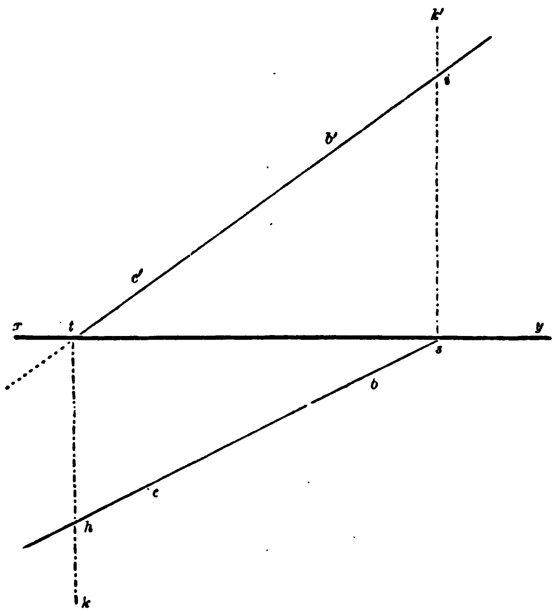
Given the projections of a straight line, to find its traces.

Let the given projections b c and b' c' (Fig. 10) meet x y in s and t respectively. The elevation of the horizontal trace will be in x y (26), and also in the elevation b' c'; it must therefore be the point t. The trace itself will be in t k drawn from t perpendicular to x y in the horizontal plane, and also in the plan b c; it must therefore be the point h, in which b c and t k intersect. In the same manner it may be shown that, if s k' be drawn from s perpendicular to x y, and cutting b' c' in i, i will be the vertical trace.

Therefore generally, to find the horizontal trace of a straight line; from the point in which the elevation meets x y, draw in

the horizontal plane a perpendicular to $x\,y$; the point in which this perpendicular meets the plan of the line will be the horizontal trace.

Fig. 10.



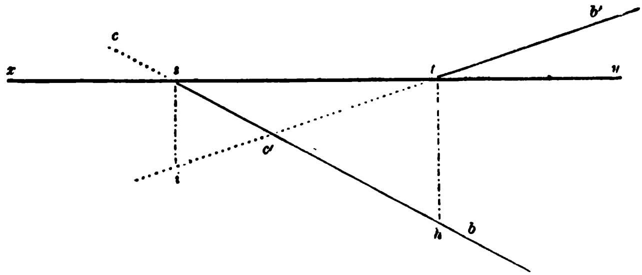
Similarly, to find the vertical trace, from the point in which the plan meets $x\,y$, draw in the vertical plane a perpendicular to $x\,y$; the point in which this perpendicular cuts the elevation will be the vertical trace.

Obs.—The portion of the straight line included between its traces may evidently have any one of the following positions, with reference to the planes of projection:—

(*a.*) It may be above the horizontal, and in front of the vertical plane (Fig. 10).

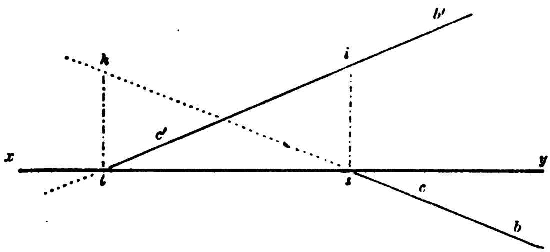
(*b.*) It may be below the horizontal, and in front of the vertical plane (Fig. 11).

Fig. 11.



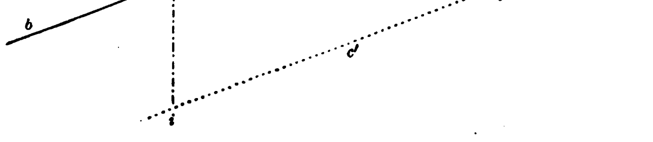
(*c.*) It may be above the horizontal, and behind the vertical plane (Fig. 12).

Fig. 12.



(*d.*) It may be below the horizontal, and behind the vertical plane (Fig. 13).

Fig. 13.



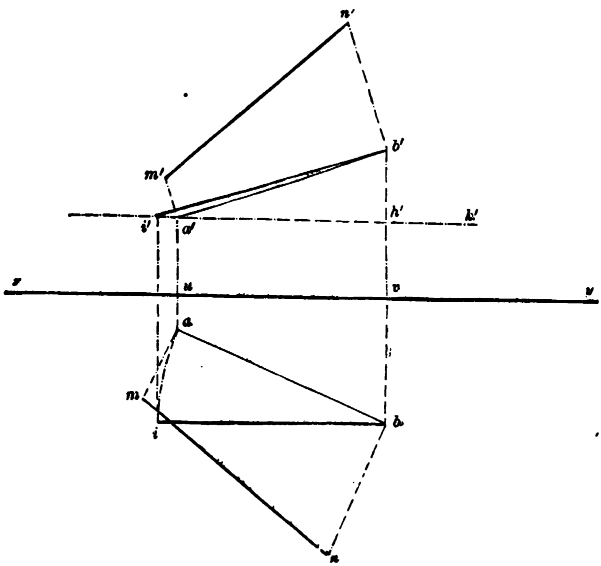
Cor. The construction of the converse Problem is obvious. For if h and i are given, the projections may be determined by drawing $h\ t$, $i\ s$ perpendicular to $x\ y$ and joining $h\ s$ and $i\ t$.

PROBLEM IV.

Given the projections of two points, to determine the length of the straight line joining the points.

Let (a, a') and (b, b') (Fig. 14) be the given points. The straight lines $a\,a'$ and $b\,b'$ will be perpendicular to $x\,y$ (30); let them cut $x\,y$ in u and v respectively. If vertical straight lines

Fig. 14.



equal to $u\,a'$ and $v\,b'$ be conceived to be drawn from a and b respectively, the extremities of these lines will be the points A and B in space, the distance between which has to be determined. Suppose a straight line drawn through the point A parallel to $a\,b$, and terminated by the vertical b B. The result will be a right-angled triangle, of which the base is equal and parallel to $a\,b$, and the perpendicular, the difference between a A and b B; that

is, the difference between $u\ a'$ and $v\ b'$; the hypothenuse being the line sought to be determined.

If therefore through a', $h'\ i'$ be drawn parallel to $x\ y$, and equal to $a\ b$, and $b'\ i'$ be joined, $b'\ i'$ will be the line required. A similar construction made in the horizontal plane will give the same result.

Otherwise. The verticals a A and b B form with $a\ b$ and A B a trapezoid, whose plane is vertical. Imagine this trapezoid turned about b B until its plane is parallel to the vertical plane of projection. The base $b\ a$ will remain in the horizontal plane, but take the position $b\ i$, parallel to $x\ y$; the line A B, in its new position, will be parallel to the vertical plane, and will consequently be projected thereon in its real magnitude (13). The point B will remain fixed; its elevation will therefore still be b'. The point A will change its position, but not its distance from the horizontal plane; its elevation will thus be in the straight line $a'\ k'$ drawn through a' parallel to $x\ y$. Now the plan of A in its new position is i; if then a straight line be drawn from i perpendicular to $x\ y$, the point i' in which $i\ i'$ cuts $a'\ k'$ will be the elevation of A when the plane a A B b is parallel to the vertical plane of projection; and the straight line $i'\ b'$ will be equal to A B.

Or, A B may be determined by turning the trapezoid a A B b about $a\ b$ until it coincides with the horizontal plane; as $a\ b\ m\ n$: or, by turning the trapezoid a' A B b' about $a'\ b'$ until it coincides with the vertical plane as $a'\ b'\ m'\ n'$.

Then, $i'\ h'=a\ b$; $a\ m=u\ a'$; $a'\ m'=u\ a$; $b\ n=v\ b'$; $b'\ n'=v\ b$.

N.B.—This problem has been discussed at length, because it is one that frequently occurs.

Problem V.

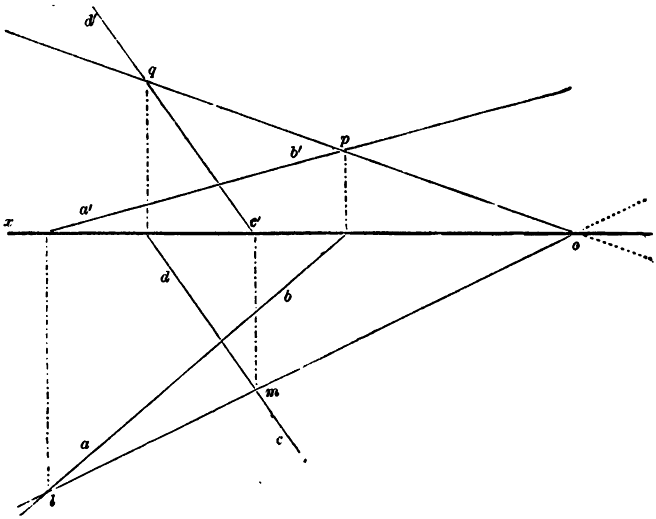
To draw a plane through two given straight lines that intersect, or are parallel.

The vertical traces of the lines will be two points in the vertical trace of the plane; the horizontal traces of the lines will be

two points in the horizontal trace of the plane. The traces are therefore determined thus:—

Let ($a\,b$, $a'\,b'$) and ($c\,d$, $c'\,d'$) (Fig. 15) be the given lines. Find by Prob. III. l and m, their horizontal traces, and p and q, their

Fig. 15.



vertical traces. The straight lines passing through p and q, and l and m respectively, will be the traces of the required plane, and should meet $x\,y$ in the same point o.

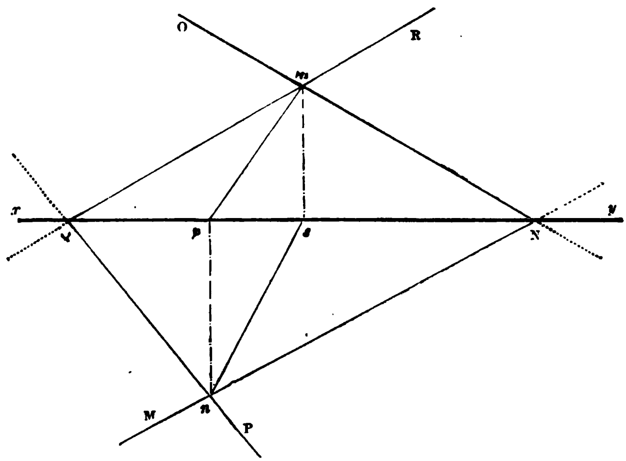
Problem VI.

To determine the projections of the intersection of two given planes.

Let M N O and P Q R (Fig. 16) be the given planes; then m, the intersection of their vertical traces, will be the vertical trace of their intersection; and n will be its horizontal trace. The intersection itself is the straight line in space joining m and n. It is required to construct the projections of this line.

Draw $m\ s$ perpendicular to $x\ y$; s will be the plan of m (27), and n is the plan of a second point in the line; consequently the

Fig. 16.



plan of the intersection is $s\ n$ (6). In the same manner it may be shown that if $n\ p$ be drawn perpendicular to $x\ y$, the elevation of the intersection is $p\ m$.

PARTICULAR CASES.

I. Let the horizontal traces M N and PQ be parallel (Fig. 17), then the intersection of the planes will be parallel to M N and to P Q; consequently the plan of the intersection will also be parallel to M N and P Q (11), and its elevation will be parallel to $x\ y$ (27). The point m, in which the vertical traces of the planes cut each other, is a point in the intersection of the planes; if therefore $m\ s$ be drawn perpendicular to $x\ y$, m and s will be the plan and elevation of m; $s\ d$ parallel to M N, and $m\ p$ parallel to $x\ y$, will therefore be the projections required.

II. Let the traces of both planes be parallel to $x\ y$, as M N, M′ N′, and P Q, P′ Q′ (Fig 18). The planes themselves will be parallel to $x\ y$, as will also their intersection.

A third plane of projection will now be necessary in order to determine the intersection. Let this plane be assumed perpen-

Fig. 17.

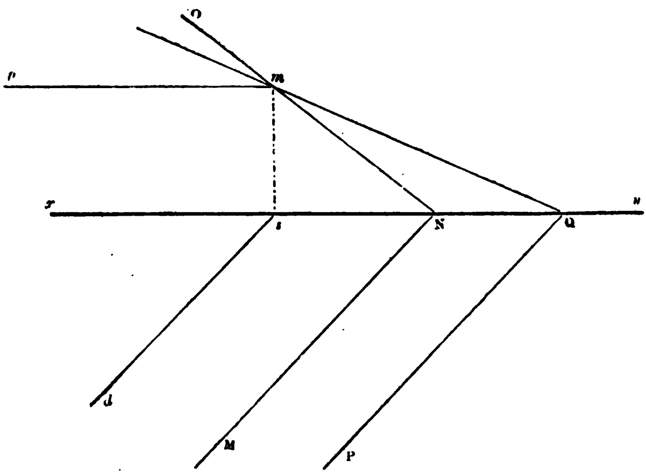
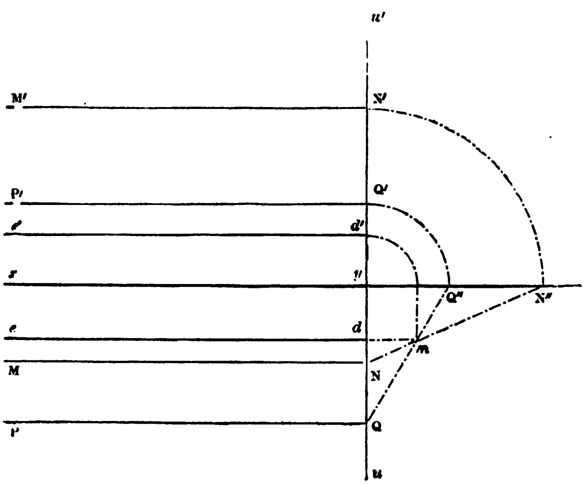


Fig. 18.



dicular to the other two, and therefore to $x\ y$.(*Euc.* XI. 19); its traces $y\ u$ and $y\ u'$ will be in one straight line perpendicular to $x\ y$ (27). Then N N″ and Q Q″, the traces of the given planes on the third plane of projection, may be found by (36): the point m, in which these traces cut each other, will be the point in which the intersection of the planes meets the third plane; d and d', the projections of m, may be found by (36), and the straight lines $d\ e$, $d'\ e'$ drawn parallel to $x\ y$ will be the projections of the intersection.

III. Let the traces of both planes meet $x\ y$ in the same point: the intersection may be determined by a construction similar to that in case II.; and the solution is left as an exercise.

IV. If the traces of one plane be parallel to those of the other, but not parallel to $x\ y$; the planes are parallel, and therefore have no intersection.

Problem VII.

To determine the point of intersection of three given planes.

The planes combined two and two cut each other in three straight lines, which all pass through the required point. Let the projections of these intersections be constructed by Prob. VI.; then if the constructions be accurately performed, the three elevations will pass through the point o' (Fig. 19); and the three plans will pass through o. The point $(o,\ o')$ will be the point required: the straight line $o\ o'$ should be perpendicular to $x\ y$ (30).

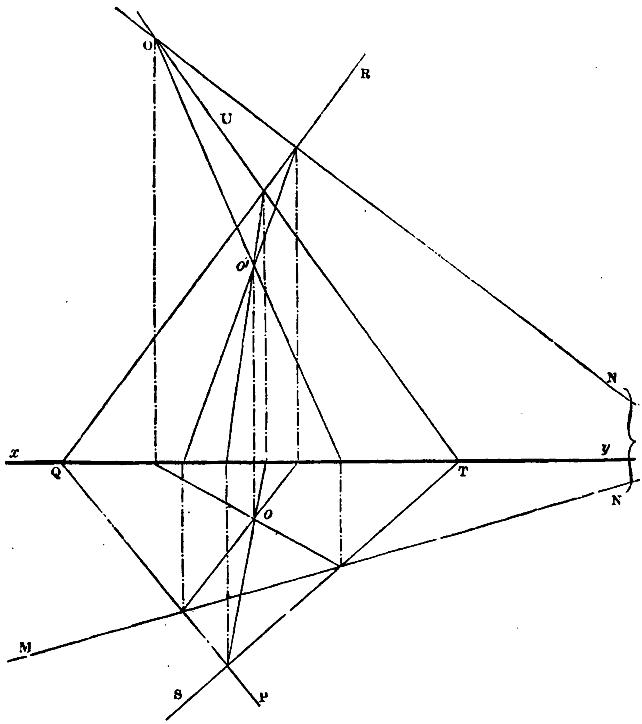
Problem VIII.

To draw a plane through three given points.

The three straight lines which join the given points, taken two and two, being situated in the required plane, will meet the planes of projection in points which are in the traces of the plane. Thus, by Prob. III. may be found three points in each

trace. The three points in each trace must be in the same straight line; and the traces must meet $x\ y$ in the same point (21).

Fig. 19.

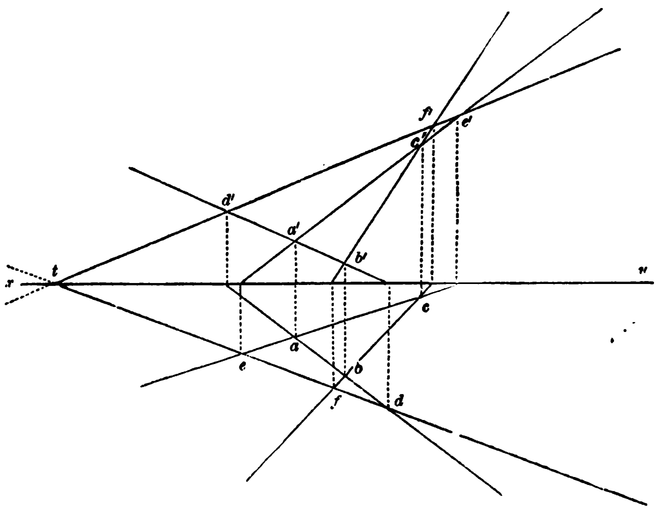


Let (a, a'), (b, b') and (c, c') (Fig. 20) be the given points: $(a\ b, a'\ b')$, $(b\ c, b'\ c')$, and $(a\ c, a'\ c')$ the lines passing through those points taken two and two. Determine (Prob. III.), d, f, e, the horizontal, and d', e', f', the vertical traces of these lines. The straight line drawn through e and f should pass through d; and that drawn through e' and f' should pass through d'; these straight lines $d\ f\ e$, $d'\ f'\ e'$, which are the required traces, must meet $x\ y$ in the same point t.

Obs.—The following cases may be considered with advantage :—

I. When one of the lines is parallel to one of the planes of projection.
II. When one of the lines is parallel to both planes of projection.
III. When two of the lines are parallel to one of the planes of projection.
IV. When the three given points are in a straight line.

Fig. 20.



Problem IX.

To determine the point in which a given straight line meets a given plane.

If, through the plan of the given line, a vertical plane be drawn, this plane will contain the point sought; but the point is also in the given plane ; it must therefore be that point in which the intersection of these two planes meets the given line.

Let $d\ n$ and $d'\ s'$ (Fig. 21) be the projections of the line; P Q R, the given plane: draw through $d\ n$ the vertical plane $d\ n\ n'$; its vertical trace $n\ n'$ will be perpendicular to $x\ y$ (26) : draw $b\ b'$ perpendicular to $x\ y$: join $b'\ n'$: $b'\ n'$ will be the

elevation of the intersection of the planes P Q R and *d n n′*. (Prob. VI.) Now the elevation of the required point will be in *b′ n′*, and also in *d′ s′*; it must therefore be the point *f′* in which

Fig. 21.

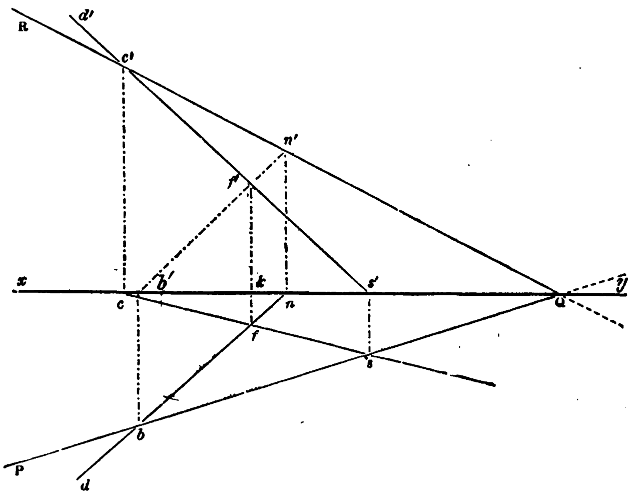
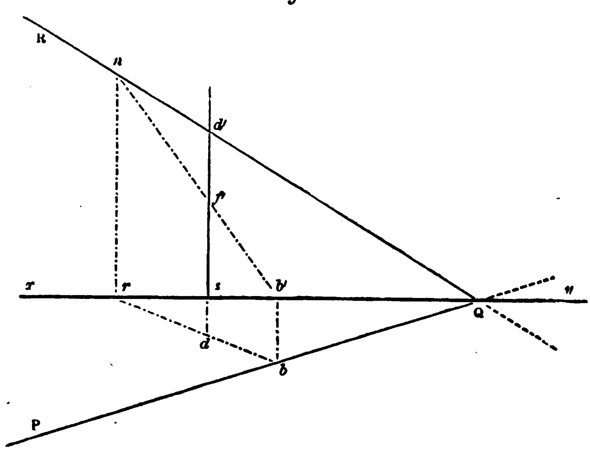


Fig. 22.



these straight lines cut each other. Through f' draw $f\,k$, per-pendicular to $x\,y$, and cutting $d\,n$ in f: f will be the plan of the required point (30). The plan f might have been determined by an independent construction, similar to that employed for the

Fig. 23.

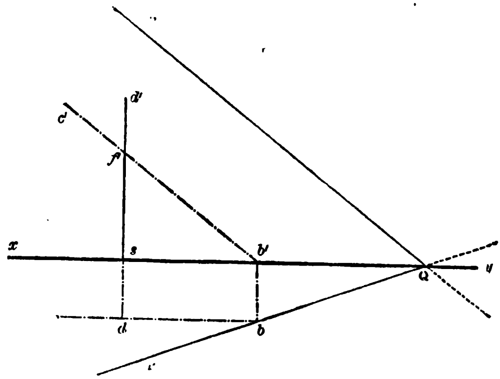
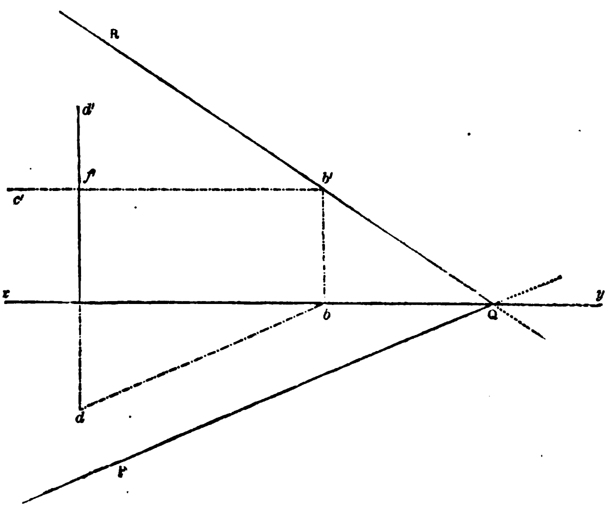


Fig. 24.



elevation, viz., by drawing through $d' s'$ a plane perpendicular to the vertical plane, as shown in Fig. 21.

If the given line be vertical, its plan will be a point (10), as d, Fig. 22; which is also the plan of the required point; the elevation of the line will be $d\,s\,d'$, perpendicular to $x\,y$ (26). To find the elevation of the point, draw through d a vertical plane, $b\,r\,n$; draw $b\,b'$ perpendicular to $x\,y$; join $b'n$, cutting $d\,d'$ in f': f' will be the elevation required.

In this case, the vertical plane assumed is subject only to the condition of passing through a given vertical straight line, it is consequently indeterminate; its trace may therefore be parallel to $x\,y$, as in Fig. 23, or to the horizontal trace of the given plane, as in Fig. 24.

PROBLEM X.

Through a given point to draw a plane parallel to a given plane.

If through the given point a straight line be drawn parallel to any straight line in the given plane, this line will be in the required plane. The traces of this line will be points in the traces of the required plane. The traces of this plane will therefore (*Euc.* XI. 16) be the two straight lines drawn through these points, and parallel respectively to the traces of the given plane.

Let (d, d') (Fig. 25) be the given point, P Q R the given plane, take any point, m, in the horizontal trace P Q; and any point n', in Q R; draw $n'\,n$ and $m\,m'$ perpendicular to $x\,y$; join $n'\,m'$ and $m\,n$, these lines will be the projections of a straight line $(m\,n, m'\,n')$ situated in the plane P Q R; because m and n' are the traces of a line in that plane (Prob. III.). Through (d, d') draw by Prob. II. a straight line parallel to $(m\,n, m'\,n')$, through f and g, the traces of this line (Prob. III.), draw M N and N O parallel to P Q and Q R respectively: M N O will be the plane required.

Solution 2. Let P Q R (Fig. 26) be the given plane: through the given point (d, d'), conceive a straight line H to be drawn parallel to P Q: this line will be in the plane required, and

will be parallel to the horizontal plane. Its plan will be the line *d r*, drawn through *d* parallel to P Q (11): its elevation will be

Fig. 25.

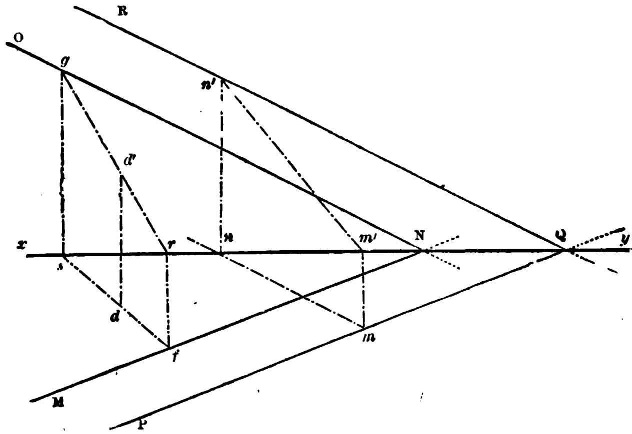
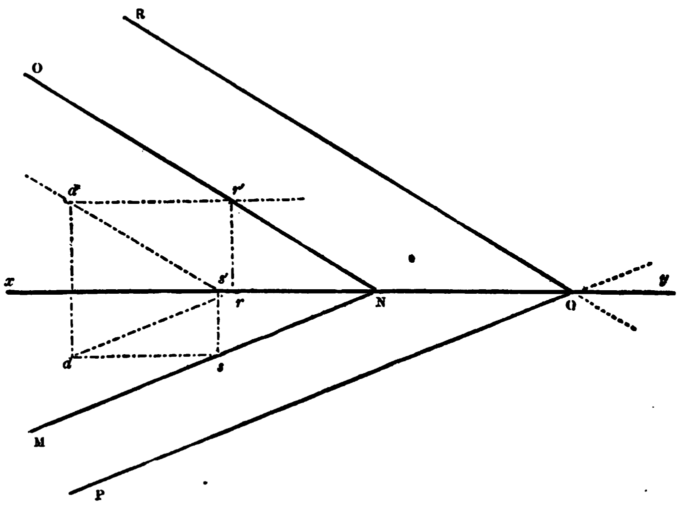


Fig. 26.



$d' r'$ parallel to $x y$ (27). The point r', in which H meets the vertical plane (Prob. III.), will be a point in the vertical trace of the plane required: the traces of which will therefore be O N drawn through r' parallel to R Q, and M N, through N parallel to P Q (*Euc.* XI. 16).

In a similar manner, the problem may be solved by drawing through (d, d') a straight line parallel to the vertical plane, as shown in the figure.

If the given plane be parallel to one of the planes of projection, as, for example, the vertical plane, it will have a horizontal trace only, which will be parallel to $x y$ (27). The required plane will also have a horizontal trace only, which will be the straight line drawn through the plan of the given point parallel to $x y$.

Problem XI.

Through a given point to draw a straight line perpendicular to a given plane: to find the point in which the perpendicular meets the plane; and to determine the length of the perpendicular.

Let (d, d') (Fig. 27) be the given point, P Q R the given plane. The projections of the perpendicular must pass through d and d' respectively: they will also be perpendicular to the traces of the plane (31). If, therefore, through d and d', $d\,n$ and $d'\,s$ be drawn perpendicular to P Q and Q R, $d\,n$ and $d'\,s$ will be the projections required. The point (p, p') in which the perpendicular meets the given plane may be determined by Prob. IX., its length, $d'\,k$, may be found by Prob. IV.

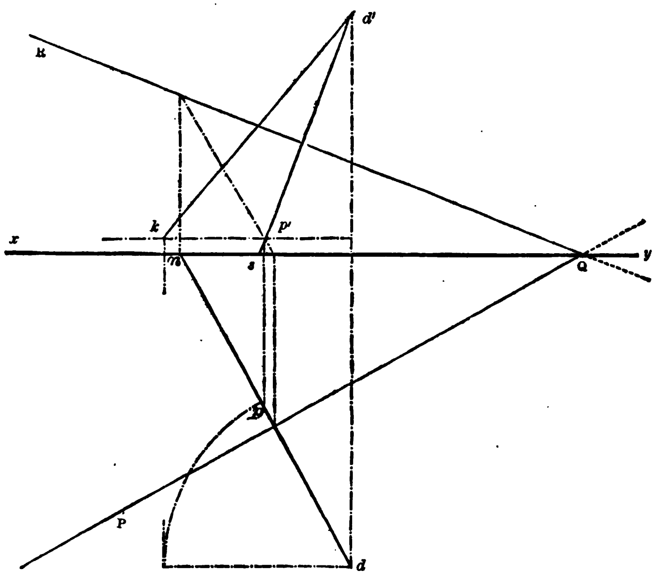
Problem XII.

Through a given point to draw a plane perpendicular to a given straight line.

Let (d, d') (Fig. 28) be the given point, $(a\,b, a'\,b')$ the given line. The traces of the required plane will be perpendicular

respectively to the projections of the given line (31). Consequently, the directions of the traces of the required plane, and one point in that plane, are known; it will therefore be sufficient if a point in one of the traces be determined. To effect this,

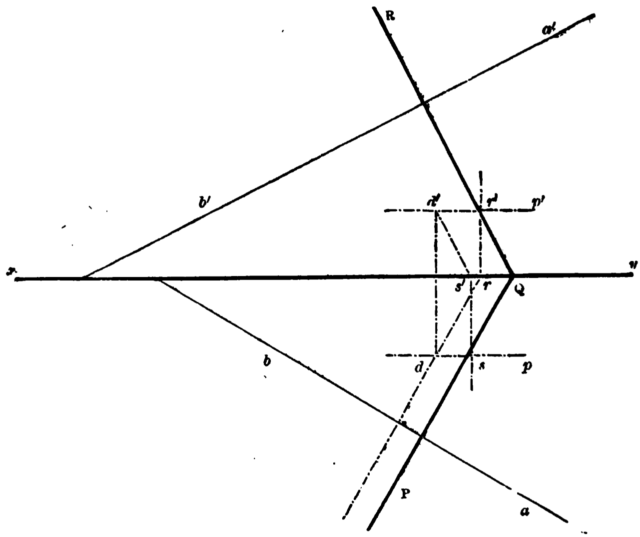
Fig. 27.



conceive a straight line to be drawn through the given point parallel to the horizontal trace of the required plane; this line will be situated in that plane, and will be parallel to the horizontal plane; its plan will therefore be parallel to the horizontal trace of the plane (11), and, consequently, perpendicular to the plan of the given line. But since this line passes through (d, d'), its plan must be $d\ r$ perpendicular to $a\ b$; its elevation $d'\ p'$ parallel to $x\ y$ (*Euc.* XI. 16). Since $d r$ and $d' p'$ are the projections of a straight line in the required plane; if r', the vertical trace of this line, be determined by Prob III., r' will be a point in the vertical trace of the required plane; and the traces

will be RQ drawn through r' perpendicular to $a'b'$; and QP drawn through Q perpendicular to ab.

Fig. 28.



A verification may be obtained by drawing through (d, d') a straight line parallel to the vertical plane, and making a construction similar to the preceding one, as shown in Fig. 28.

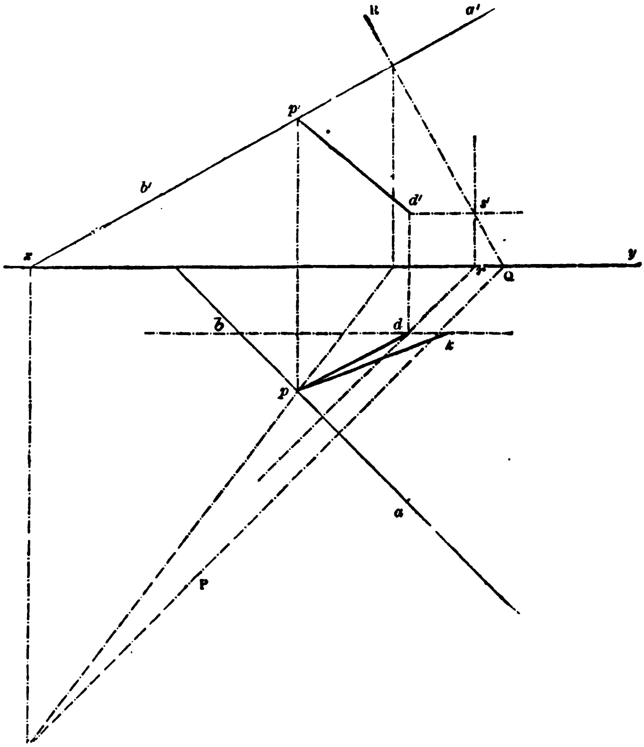
Problem XIII.

From a given point to draw a straight line perpendicular to a given straight line, and to determine the length of the perpendicular.

Let (d, d') (Fig. 29) be the given point, $(ab, a'b')$ the given line. Through (d, d') draw a plane PQR perpendicular to $(ab, a'b')$ (Prob. XII.); determine (Prob. IX.) the point (p, p')

in which the plane P Q R meets the given line; then $(d\,p, d'\,p')$ will be the perpendicular required; its length, $p\;k$, may be determined by Prob. IV.

Fig. 29.



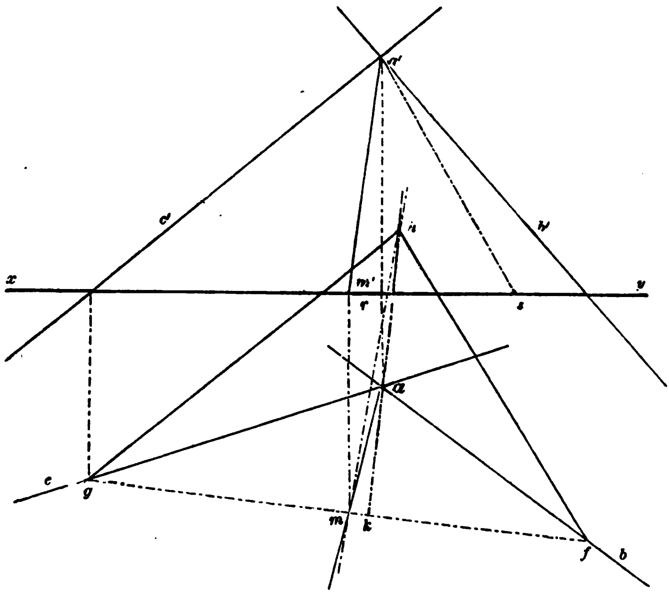
Problem XIV.

Given the projections of two straight lines which cut each other, to determine the angle contained by the lines.

Let $(a\,b, a'\,b')$ and $(a\,c, a'\,c')$ (Fig. 30) be the given lines. Determine their horizontal traces g and f (Prob. III.); join $g\,f$;

gf and the portions of the straight lines intercepted between their point of intersection and their horizontal traces, form a triangle in which the angle opposite to *gf* is the angle to be determined. To construct this triangle, through *a* draw *h k* perpendicular to *gf*; conceive the plane of this triangle to be turned about its

Fig. 30.



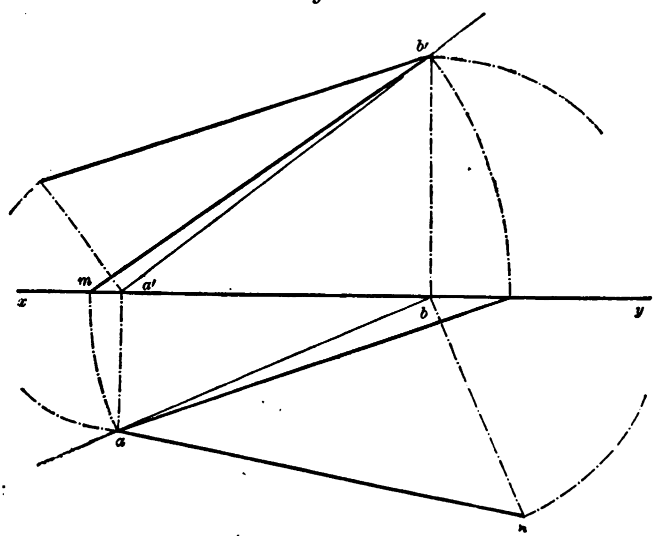
horizontal trace *gf*, until it coincides with the horizontal plane *h*, the position which the point of intersection of the lines now takes, is found (36) thus, join *a a'*, let it cut *x y* in *r*; make *r s* equal to *ak*; join *a' s*, make *h k* equal to *a' s*; join *gh* and *fh*; *ghf* will evidently be the angle contained by the lines.

PROBLEM XV.

Given the projections of a straight line to determine the angles which it makes with the planes of projection.

Let (*ab*, *a′b′*) (Fig. 31) be the given line; find its traces *a* and *b′* (Prob. III.). Then the angle which the given line makes with the horizontal plane is the angle contained by the straight line in space drawn from *a* to *b′*, and the plan *ab* (*Euc.* XI. *def.* 5). If,

Fig 31.



therefore, the plane *abb′* be turned about *ab* until it takes the position *abn* in the horizontal plane; *bn* being equal to *bb′*; *ban* will be the angle required. The angle may also be determined by turning the plane *abb′* about *bb′* until it takes the position *b′bm* in the vertical plane, when *b′mb* will be the required angle. The inclination of the line to the vertical plane may be found by a similar construction, as shown in the figure.

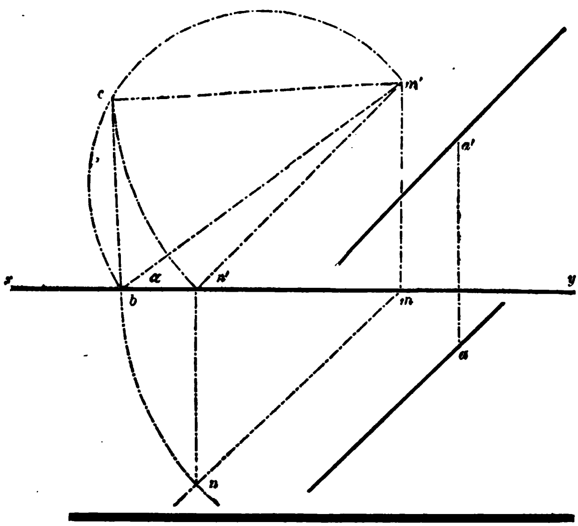
Conversely. Through a given point to draw a straight line, making given angles with the planes of projection.

This problem will be most easily solved by assuming a point on the vertical plane, and drawing through it a line having the given inclinations, and then drawing through the given point a line parallel to this line.

Let α and β be the angles which the line is to make with the horizontal and vertical planes respectively, it is evident that these are the angles between it and its projections.

Assume any point (m, m') (Fig. 31a) in the vertical plane; make the angle $m'\,b\,m$ equal to α; with centre m, and radius $m\,b$, de-

Fig. 31*a.*



scribe the circle $b\,n$; all lines passing through (m, m') and making an angle α with the horizontal plane, will have their horizontal traces in the circumference of this circle (34, *Cor.* 1). Again, on $m'\,b$ describe a right-angled triangle, having the angle $b\,m'\,c$ equal to β; make $m'\,n'$ equal to $m'\,c$ (the construction is shown in the figure), draw $n'\,n$ perpendicular to $x\,y$; n will be the

horizontal trace of the line. Its projections will evidently be $m' n'$ and $m n$.

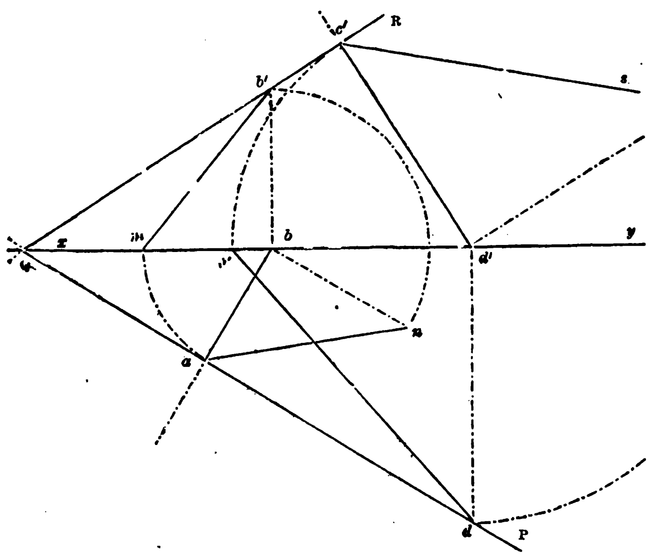
Let (a, a') be the given point; draw through a and a' parallels to $m n$ and $m' n'$ respectively, these will be the projections of the line required (11).

Problem XVI.

Given the traces of a plane, to determine the angles which it makes with the planes of projection.

Let P Q R (Fig. 32) be the given plane. To determine the angle which it makes with the horizontal plane, take any point a

Fig. 32.



in the horizontal trace P Q; through a draw a plane $a b b'$ perpendicular to the given plane and the horizontal plane; the profile angle of the given plane and the horizontal plane will

evidently be the angle required. The projections of the lines containing this angle are known, and thus the angle $b'mb$ may be determined by Problem XV. as shown in Fig. 32.

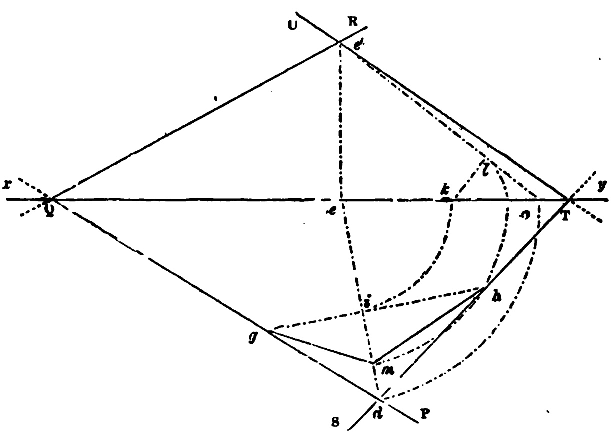
In a similar manner the inclination of the plane to the vertical, viz., the angle $d'rd$, may be found (Fig. 32).

Problem XVII.

To determine the angle contained by two given planes.

Let PQR and STU (Fig. 33) be the given planes. Find, by Prob. VI., de the plan of their intersection; let gh be the horizontal trace of a profile plane, of the planes PQR and STU,

Fig. 33.



cutting de in i. Then, since this plane is perpendicular to the intersection of PQR and STU, gh cuts de at right angles (31), it also forms, with the straight lines in which the profile plane cuts PQR and STU, a triangle, in which the angle opposite to gh is the angle sought to be determined.

Conceive the plane of this triangle to be turned about gh until it coincides with the horizontal plane; the vertex of the triangle

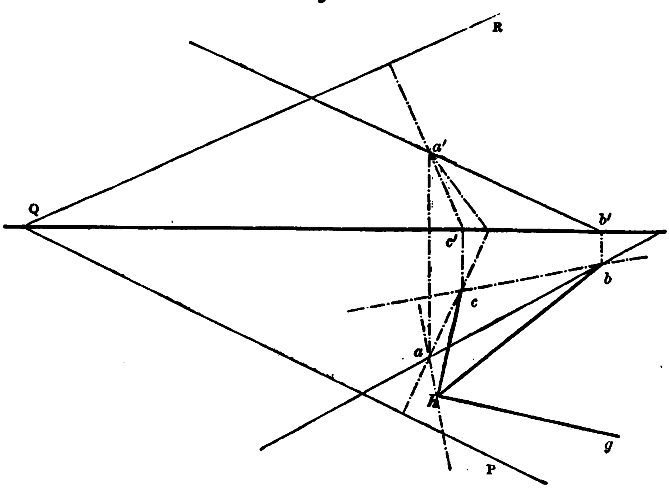
will then be in ed (36). It only remains, therefore, to determine the altitude of the triangle, that is, the straight line drawn from i perpendicular to the intersection of the planes. This perpendicular evidently lies in the vertical plane dee'; turn this plane about its trace ee' until it coincides with the vertical plane; make ek equal to ei; draw kl perpendicular to $e'o$; kl will be the altitude of the triangle required. If, then, im be made equal to kl, and gm, hm be drawn, the angle gmh will be the angle contained by the planes. This angle may be determined by a similar construction in the vertical plane.

Problem XVIII.

To determine the angle contained by a given straight line and a given plane.

Let PQR (Fig. 34) be the given plane, $(a\,b, a'b')$ the given line. From any point (a, a') in the line, draw a straight line

Fig. 34.



perpendicular to the given plane (Prob. XI.); the projections of this perpendicular will be $a\,c$ and $a'\,c'$, perpendicular respectively

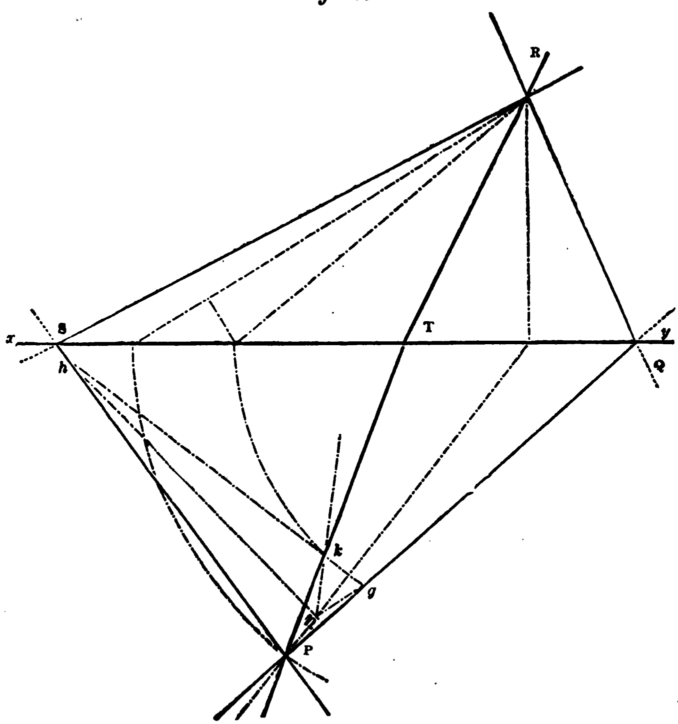
to P Q and Q R the traces of the plane (31). The angle contained by this perpendicular and the given line will be the complement of the required angle; this angle between the perpendicular and the given line is found by Problem XIV. to be *b h c*; if, therefore, *h g* be drawn perpendicular to *h c*, *b h g* will be the angle required.

Problem XIX.

To draw a plane bisecting the angle between two given planes.

Let P Q R and P S R (Fig. 35) be the given planes; *h n g* the angle between them, determined by Problem XVII. Bisect the

Fig. 35.



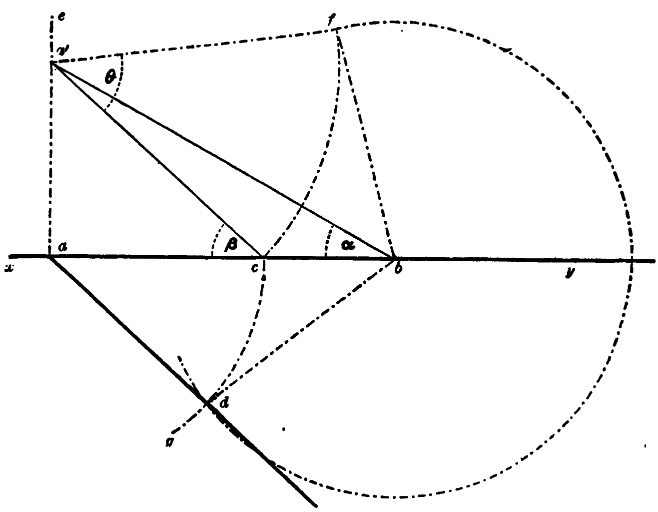
angle $h\,n\,g$ by the line $n\,k$, meeting $h\,g$ in k; k will be a point in the horizontal trace of the plane required. But P is also a point in that trace; the traces of the plane bisecting the angle will therefore be P T passing through k, and T R passing through R, since R is a point in the vertical trace.

Problem XX.

To reduce an angle to the horizon. That is, having given the angle which two straight lines make with each other, and their inclinations to the horizontal plane, to determine the plan of the angle.

Let a (Fig. 36) be the plan of the vertex of the angle: $a\,b$ that of one of the lines containing it, and inclined at an angle

Fig. 36.



α to the horizontal plane; also, let the inclination of the second line to the horizontal plane be β; θ being the angle contained by the lines.

Assume the vertical plane of projection to pass through the first line, so that $a\ b$ coincides with $x\ y$: from a draw $a\ e$ perpendicular to $a\ b$; the angular point will be in $a\ e$: let it be a': through a' draw $a'\ b$, making an angle α with $a\ b$: b will be the horizontal trace of the first line. Through a' draw $a'\ c$, making an angle β with $a\ b$: with a as a centre and radius $a\ c$ describe the circle $c\ d\ g$: all straight lines passing through a' and making an angle β with the horizontal plane, will have their horizontal traces in the circumference of this circle (34 *Cor.* 1). Again, make the angle $b\ a'\ f$ equal to θ; and $a'\ f$ equal to $a'\ c$: join $b\ f$; with b as a centre and radius $b\ f$ describe a circle cutting the circle $c\ d\ g$ in d; draw $a\ d$: $a d$ will be the plan of the second line.

For the straight line in space joining a' and d, makes an angle β with the horizontal plane: it also makes an angle θ with $a'\ b$: for in the triangles $a'\ d\ b$ and $a'\ f b$, $d\ a'$ and $a'\ b$ are respectively equal to $f\ a'$ and $a'\ b$; the base $b\ d$ is equal to the base $b\ f$; therefore the angle $d\ a'\ b$ is equal to the angle $b\ a'\ f$ (*Euc.* I. 8), which is equal to θ. The line $a'\ d$ thus fulfils the conditions of the problem, and its plan is $a\ d$: therefore the angle $d\ a\ b$ is the plan of the angle required.

Problem XXI.

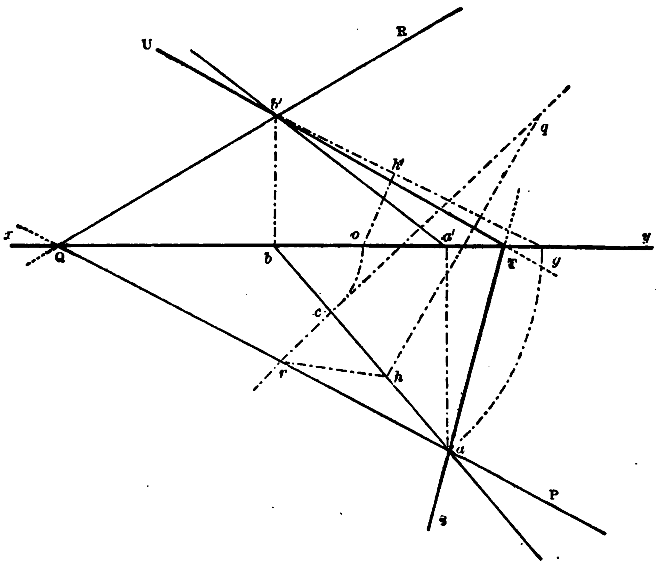
Through a straight line, in a plane, to draw a plane making a given angle with that plane.

The straight line will evidently be the intersection of the given plane and the required plane.

Let $(a\ b,\ a'\ b')$ (Fig. 37) be the given line, P Q R the given plane: draw a straight line $q\ r$ perpendicular to $a\ b$, and cutting it in c: $q\ r$ will be the horizontal trace of a profile plane of the given and required planes (31); and the lines in which this plane cuts the two planes will form the angle which those two planes make with each other (33). The vertex of this angle will be in the vertical plane passing through $a\ b$, and in the straight line drawn from the point c in $q\ r$ perpendicular to the line $(a\ b,\ a'\ b')$. Turn the plane $a\ b\ b'$ about $b\ b'$ until it coincides with

the vertical plane of projection: the line ($a\ b$, $a'\ b'$) and the point c, will take the positions $g\ b'$ and o respectively. Draw $o\ h'$ perpendicular to $g\ b'$; set off $c\ h$, in $c\ a$, equal to $o\ h'$: make the angle

Fig. 37.



$r\ h\ q$ equal to the given angle: q will be the horizontal trace of a line in the plane required, and consequently a point in the horizontal trace of the plane itself. The required traces will therefore be S a T q and U b' T : S T U being the plane required.

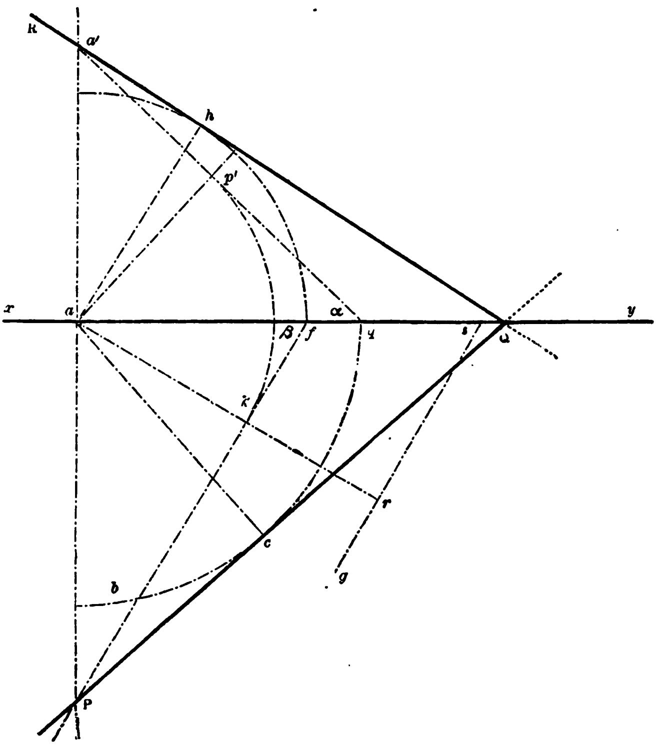
Problem XXII.

Through a given point to draw a plane making given angles with the planes of projection.

Let the given point (a, a') (Fig. 38), be assumed in the vertical plane of projection: α and β being the angles which the required

plane is to make with the horizontal and vertical planes respectively. Make the angle *a′ q a* equal to *α*: with centre *a* and radius *a q* describe a circle *q c b*: the horizontal trace of the plane will be a tangent to this circle. Draw the line *g s*, making with *x y* the angle *g s x* equal to *β*: draw *a p′* perpendicular to

Fig. 38.



a′ q; *a r* perpendicular to *g s*, and make *a k* equal to *a p′*: through *k* draw P *k f*, parallel to *g s* and meeting *x y* in *f*, *a′ a* produced, in P: with centre *a* and radius *a f* describe the circle *f h*: the vertical trace of the required plane will be a tangent to this circle. Through P draw P Q, touching the circle *b c q* in *c*:

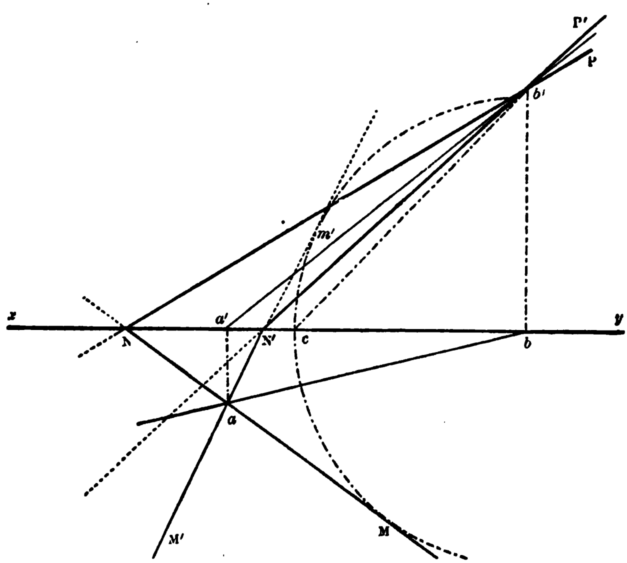
through Q draw Q R, touching the circle *f h* in *h*. P Q R will be the plane required. The solution of this problem, which is indeterminate, will be easily understood by referring to Prob. XVI. The lines *a p′*, *a k* are the perpendicular drawn from the point *a* to the plane.

Problem XXIII.

Through a given straight line to draw a plane, making a given angle with the horizontal plane.

Find, by Prob. III., the traces *a* and *b′* (Fig. 39) of the given line (*a b*, *a′ b′*). These will be points in the traces of the required

Fig. 39.



plane. It will therefore be sufficient if a second point in one of the traces be determined. To find a point in the horizontal trace, at *b′* in *b b′* make the angle *b b′ c* equal to the complement of the given angle, with *b* as a centre and radius *b c* describe the circle *c* M. If this circle cuts *a b*, the tangents *a* M and *a m′* drawn

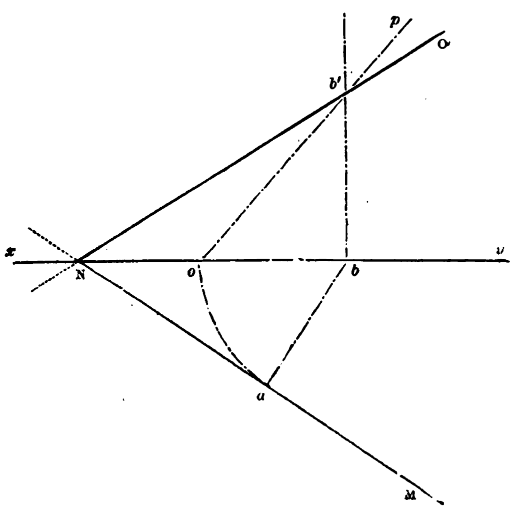
to it from a, will be the horizontal traces of two planes fulfilling the conditions of the problem. If the circle passes through a, only one such plane can be drawn: because the inclination of the plane will be equal to that of the given line. If a falls within the circle the problem is impossible, since the inclination of the plane would be less than that of the line (34). The planes, as determined in Fig. 39, will be M N P and M′ N′ P′ since the vertical traces must pass through b'.

Problem XXIV.

To find the vertical trace of a given plane, when its horizontal trace and inclination are given.

Let M N (Fig. 40) be the given trace; α the angle at which the plane is inclined to the horizontal plane. Through any point

Fig. 40.



a in M N draw a vertical plane a b b', perpendicular to M N; turn this plane about b b' until it coincides with the vertical

plane of projection: the point a will take the position o; $b\,o$ being equal to $b\,a$: make the angle $b\,o\,p$ equal to α: b' the point in which $o\,p$ cuts $b\,b'$ will be a point in the vertical trace of the plane: through b' draw N O; this will be the vertical trace.

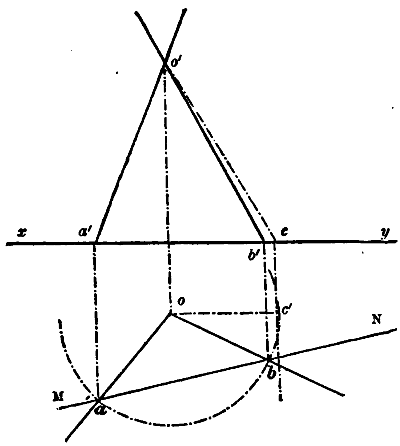
Should the trace M N not meet $x\,y$ within the limits of the drawing, a second point in the vertical trace must be determined in the same manner as b'.

Problem XXV.

Through a given point to draw a straight line which shall make a given angle with the horizontal plane, and meet a given straight line in that plane.

Let (o, o') (Fig. 41) be the given point, M N the given line situated in the horizontal plane. At o' in $o\,o'$, make the angle

Fig. 41.



$o\,o'\,c$ equal to the complement of the given angle; draw $c\,c'$ perpendicular to $x\,y$; and $o\,c'$ perpendicular to $c\,c'$. With o as a centre, and radius $o\,c'$, describe a circle cutting M N in a and b; draw $a\,a'$ and $b\,b'$ perpendicular to $x\,y$; join $o'\,a'$ and $o'\,b'$; the

two lines ($o\ a$, $o'\ a'$) and ($o\ b$, $o'\ b'$) will evidently fulfil the conditions of the problem (34 *Cor.* I).

This problem depending upon the intersection of a straight line and a circle, will have one solution or two solutions, or will be impossible.

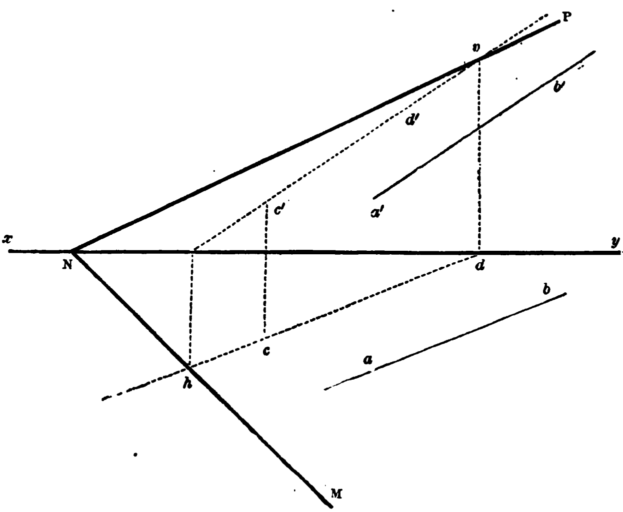
Problem XXVI.

Through a given point to draw a plane parallel to a given straight line.

Let (c, c') (Fig. 42) be the given point, ($a\ b$, $a'\ b'$) the given line.

Through (c, c') draw a straight line ($c\ d$, $c'\ d'$) parallel to ($a\ b$, $a'\ b'$) (Prob. II.); determine its traces h and v (Prob. III.). Any

Fig. 42.



plane, M N P, whose traces pass through h and v respectively, will be parallel to ($a\ b$, $a'\ b'$). The problem is therefore indeterminate.

Problem XXVII.

To draw a straight line bisecting the angle between two given straight lines.

The straight line bisecting the angle between the two lines lies in the plane of these lines, and passes through their point of intersection. Determine the angle contained by the lines (Prob. XIV.); bisect it: the point in which the bisecting line meets the straight line joining the horizontal traces, will be a point in the plan of the bisecting line; this point and the projections of the angular point are sufficient to determine the projections required. The construction may be seen in Fig. 30, where ($a\,m$, $a'\,m'$) is the bisecting line.

Problem XXVIII.

Given the three sides of a spherical triangle, to determine the angles.

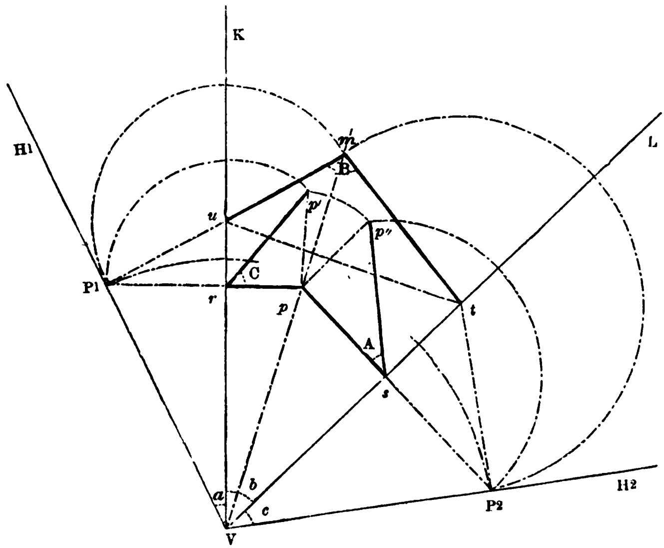
Let a, b, c be the given sides; A, B, C the corresponding angles required to be determined.

Suppose the faces of the angles a and c to be turned about the edges VK, VL (Fig. 43) until they coincide with the plane of the side b as shown in the figure, in which VH_1, VH_2 are the new positions of the third edge: take VP_1 equal to VP_2, and draw P_2s perpendicular to VL, P_1r perpendicular to VK: if P_2s and P_1r meet in p, p will be the projection of the point P on the plane of the side b (by the point P is meant the point which takes the positions P_1, P_2 after the faces containing it have been turned down). Draw pp' perpendicular to rp, with r as a centre and radius rP_1 describe a circle cutting pp' in p': join $p'r$, the angle prp' will be the angle C measuring the inclination of the faces of the sides a and b as is evident from (36), since prp' is the profile plane of these faces. The angle A may be found by a similar construction.

To determine the third angle B, draw P_1u and P_2t perpen-

dicular to VH_1 and VH_2 respectively, and meeting VK and VL in u and t; join ut; ut will be the base of the triangle formed by the intersections of the three faces, by the profile plane of the faces of a and c: the other sides of this triangle will evidently be equal to P_1u and P_2t: if therefore with centres u and t, and

Fig. 43.



radii uP_1 and tP_2, circles be described intersecting in m: and um, tm be joined, the angle umt will be equal to the third angle B.

The line Vm being the projection of the third edge upon the plane of the face KVL, the points V, p, m must be in a straight line; it is also evident that the perpendiculars pp' and pp'' are equal to each other since each of them is the distance of the point P from the plane KVL.

Conversely. Given the three angles of a spherical triangle to determine the sides.

This case can be reduced to the preceding one by means of the polar triangle.

Let A, B, C be the given angles, then the sides of the polar

triangle are $180° - A$, $180° - B$, $180° - C$: determine the angles of this triangle, then if they are A', B', C': the sides required will be $a = 180° - A'$, $b = 180° - B$, $c = 180° - C'$.

Obs. The solution of the problem will be possible only when

$$a + b + c > 2\pi, \text{ and } a + b < c,\ a + c > b,\ b + c > a.$$

In the converse $A + B + C$ must be $> \pi$ and $< 3\pi$ in order that the solution may be possible.

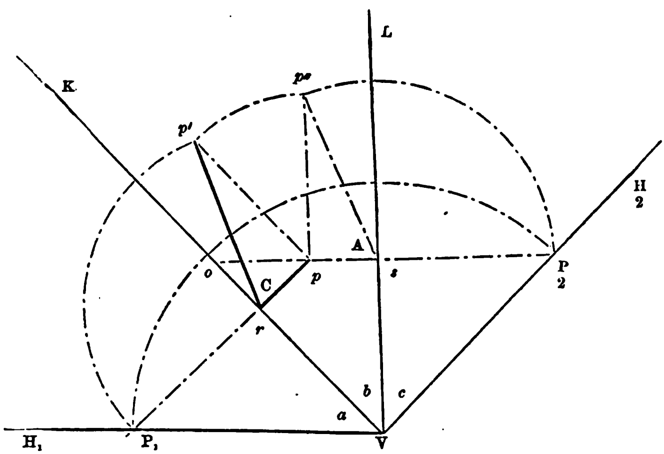
Problem XXIX.

Given two sides of a spherical triangle and the included angle, to find the remaining parts.

Let the sides b, c and the angle A be given.

Let H_2VL (Fig. 44) be the face of the side c turned about the edge VL into the plane of the side b; from any point P_2

Fig. 44.



in VH_2, draw P_2p perpendicular to VL and cutting it in s, make the angle psp'' equal to the given angle A, make $sp'' = sP_2$: draw $p''p$ perpendicular to sp: p will be the projection of the point P

on the plane of the side b (Chap. I. 36); draw pP perpendicular to VK$_1$ and cutting it in r; draw pp' perpendicular to pP$_1$ and equal to pp''; join rp': the angle prp' will be equal to the angle C. Again make $r\mathrm{P}_1=rp'$; through P, draw VH$_1$; the angle KVH$_1$ will evidently be the third side a. If Ps produced meets VK in o; $p''o$ is the real length of the intersection of the face of the side a by the profile plane of the faces of the sides b and c; the point P may therefore be determined by making $o\mathrm{P}_1=op''$, without determining the angle C.

The angle B may be found by Prob. XXVIII.

Conversely. Given B, C and a, to find b, c and A.

The two corresponding sides of the polar triangle are $180°-\mathrm{B}$, $180°-\mathrm{C}$, and the included angle $180°-a$: solve this triangle, then if its two angles are B$'$, C$'$ its third side a', $b=180°-\mathrm{B}'$, $c=180°-\mathrm{C}'$ and $\mathrm{A}=180°-a'$.

Problem XXX.

Given two sides of a spherical triangle and the angle opposite to one of them, to find the remaining parts.

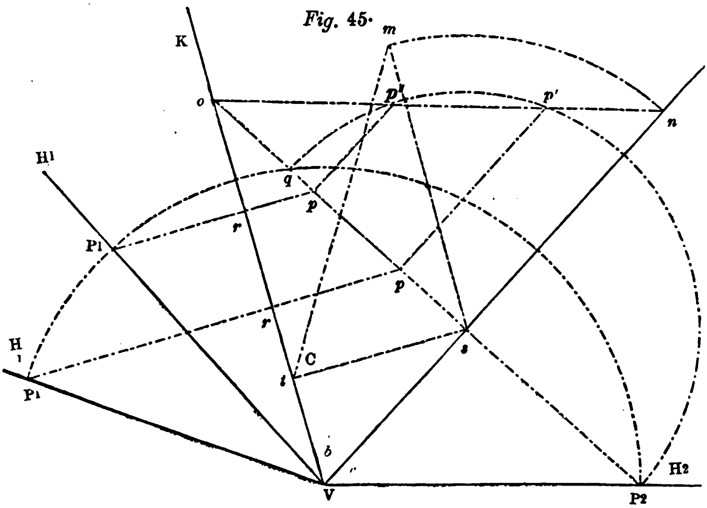
Let the sides b, c, and the angle C be given.

Let the plane of the side c be turned about the edge VL (Fig. 45) until it coincides with the plane of the side b. From t, a point in the edge VK, draw ts perpendicular to VK; make the angle stm equal to the angle C and draw sm perpendicular to st: if the triangle mts be turned about st until its plane is perpendicular to that of the side b, m will be a point in the plane of the side a; draw P$_2o$ perpendicular to the edge VL and cutting it in s, conceive this to be the trace of a profile plane of the planes of b and c; this plane will cut the plane of the side a in a straight line passing through m: if therefore sn be taken equal to sm, and on be joined, on will be the real length of this intersection, as will be seen by conceiving the triangle ons to revolve about so until its plane is perpendicular to the face KVL. Now the point in which the edge VP meets this plane must be in on;

it must also be in the semicircle of which s is the centre, and sP_2 the radius, since sP_2 is the true length of the section of the face P_2VL by the profile plane. If therefore the circle cuts *on* in two points the problem admits of two solutions; if it touches *on* there is only one solution; if it does not meet *on* there is no solution.

Let p' be a point in which the circle $P_2pp'q$ meets *on*; draw pp' perpendicular to P_2o, and pP_1, perpendicular to VK, with o as a centre and radius op' describe a circle cutting pP_1, in P_1; P_1

Fig. 45.



will be a point in the third edge when its plane is turned into coincidence with that of the side b, since op' is the real length of the line in which the third face is cut by the profile plane P_2on. Through P_1 draw VH_1, the angle KVH_1 will be the third side of the triangle.

The angles B and C may be found by Prob. XXVIII.

Conversely. Given B, C and c to find A, a and b.

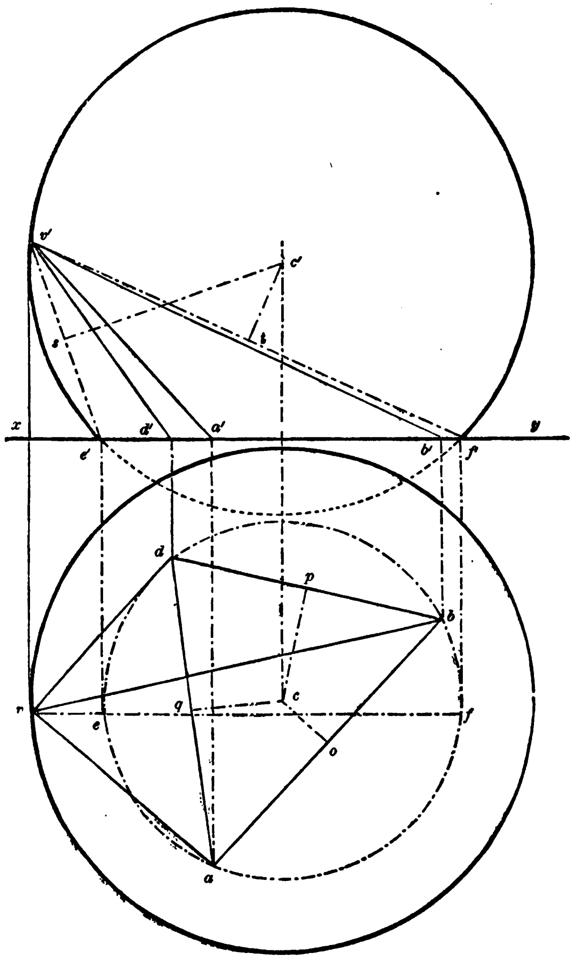
Here two sides of the polar triangle are $180°-B$, $180°-C$, and the angle opposite to the latter is $180°-c$, whence the third side and the angles may be found as above, let these be a', A' and B' then $A=180°-a'$, $a=180°-A'$, $b=180°-B'$.

Problem XXXI.

To circumscribe a sphere about a triangular pyramid.

Let *abdv* (Fig. 46) be the plan of the solid, assuming the plane of the face *abd* as the horizontal plane, *v′b′a′d′* its elevation.

Fig. 46.



Let c be the centre of the circle abd, circumscribing the triangle abd, c will be plan of the centre of the sphere, its elevation will be some point in the line cc' perpendicular to xy. Through v draw vef parallel to xy, let $e'f'$ be the elevation of ef; join $v'e'$, $v'f'$; these are the elevations of the lines ve and vf, as well as the real lengths of the lines, since they are parallel to the vertical plane. Now it is evident that all the points in the circumference of the circle abd are on the circumference of the sphere, therefore $(ve, v'e')$ being a chord of the sphere the elevation of the centre must be in sc' drawn from the middle point of sc' perpendicular to it, it will therefore be the point c' in which sc' meets cc': the radius being the straight line $(ca, c'a')$ is easily found to be $c'e'$ (Chap. II. Prob. IV.). The projections will therefore be the circles described with c' and c as centres and radius $c'e'$.

Problem XXXII.

Through a given point to draw a straight line parallel to a given plane.

This problem is indeterminate, since a straight line drawn parallel to any line in the plane will fulfil the condition (Prob. II).

Problem XXXIII.

Through a given point to draw a straight line parallel to two given planes.

Construct the intersection of the planes (Prob. VI.); and through the given point draw a straight line parallel to the intersection (Prob. II.). This will be the straight line required.

Problem XXXIV.

Through one given straight line to draw a plane parallel to another given straight line.

Through any point in the first line, draw a line parallel to the second (Prob. II.); then by Prob. V. draw a plane containing

this parallel and the first line. This will be the plane required. This problem is always possible; but it will be indeterminate when the two given lines are parallel, for in that case every plane passing through the first line, and not through the second, will be parallel to the second.

Problem XXXV.

Through a point in one given plane to draw a straight line parallel to another given plane.

Construct the intersection of the planes, (Prob. VI.,) and through the given point draw a straight line parallel to the intersection. This will be the line required.

Problem XXXVI.

Through a given point to draw a plane perpendicular to two given planes.

Find the intersection of the given planes (Prob. VI.); draw through the given point a plane perpendicular to this intersection (Prob. XII.). This will be the plane required (*Euc.* XI. 18).

Problem XXXVII.

Through a given straight line to draw a plane perpendicular to a given plane.

From any point in the straight line, draw a straight line perpendicular to the given plane (Prob. XI.). The plane containing this perpendicular, and the given line, will be the plane required (Prob. V., *Euc.* XI. 18).

Problem XXXVIII.

Given the plan of a polygon situated in a given plane; to find its elevation and real magnitude.

Let *abcde* (Pl. I. Fig. 2) be the plan of a polygon, situated in the given plane P Q R. The elevations of the angular points

of the polygon will be found by constructing the elevations of the points in which vertical lines drawn through a, b, c, d and e, respectively, meet the plane P Q R. These elevations may be constructed by Problem IX. Let them be a', b', c', d', e'. The figure $a'\ b'\ c'\ d'\ e'$ will be the elevation required. The angular points of the polygon A B C D E itself are found by turning the plane P Q R about the trace P Q in the manner explained in (36, III.).

Problem XXXIX.

To determine the projections of a polygon situated in a given plane.

Let A B C D E (Pl. I. Fig. 2) be the given polygon; P Q R the given plane. The point a, which is the plan of A, is situated in A a, drawn from A perpendicular to P Q and at a distance from P Q equal to $o''a$, the base of the right-angled triangle $o''a\,a''$; in which triangle the angle $a''o''a$ is equal to the angle $n\,o''o'$, the inclination of the plane P Q R to the horizontal plane (Prob. XVI.); the hypothenuse $a''o''$ is equal to A m (36). In the same way b, c, d, and e may be found, and the plan completed by joining the respective points. Again, the side $a\,a''$ is the height of the point A above the horizontal plane. If, therefore, $a\,a'$ be drawn perpendicular to $x\,y$, and $a''\,a'$ parallel to $x\,y$, the point a', in which these lines meet, will be the elevation of A. Similarly, b', c', d', and e' may be found, and the elevation completed.

Problem XL.

To determine the projections of a circle situated in a given plane.

1. Let the plane of the circle be parallel to one of the planes of projection, and perpendicular to the other. Its projection on the first will be a circle equal to the original one (13): its projection upon the second will be a straight line equal to the diameter of the circle (12 and 13).

2. Let the plane of the circle be inclined at any angle to the planes of projection.

Let A C D F H (Pl. I. Fig. 3) be the circle; P Q R the plane in which it is situated. Divide the circumference into equal parts in the points A, B, C, D, E, F, G, H: determine a, b, c, d, e, f, g, h, the plans, and $a', b', c', d', e', f', h'$, the elevations of these points as in Prob. XXXVIII. by (36). The curves $a\,b\,c\,d\,e\,f\,g\,h$ and $a'b'c'd'e'f'g'h'$ will be the projections required.

These curves will be ellipses, whose major axes, being the projections of diameters parallel respectively to the planes of projection, are equal to the diameter of the circle.

Exercises.

1. The plan of a line is 1·75 in. long, and the difference of level of its ends 1·2 in. Find the length of the line and its inclination.

2. Represent by their plans and elevations two points A and B: A, 2 in. above the horizontal plane, and 1·5 in. in front of the vertical plane; B, 1·75 in. below the H. P. and 1 in. behind the V. P., the distance between the plans being 3·25 in. Find the length and inclination of the line A B.

3. The horizontal projection of a line is 6 ft. long and inclined to the axis at an angle of 30°; the vertical projection meets the axis at an angle of 20°. Construct the length of the line on a scale of $\frac{1}{20}$.

4. Two points a and b are 30 units apart, and the lines $a\,b$, $a'\,b'$ make respectively angles of 25° and 30° with $x\,y$; determine the line A B.

5. The elevations of two lines intersect at an angle of 80°, their plans, at an angle of 100°: find the angle between the lines.

6. Determine by their traces two parallel planes 1·5 in. apart, the horizontal traces not being parallel to $x\,y$.

7. Show how to draw a straight line from a given point perpendicular to a given plane.

Draw from the point (3·5 in., 2·7 in.) a straight line perpendicular to a plane inclined at 50° to the H. P.: and not perpendicular to the V. P.

8. A line inclined at 35° lies in a plane inclined at 40°, draw a plane inclined at 50° and containing the line. Determine the angle betweeen the planes.

9. Through the line in Ex. 8 draw a plane making an angle of 54° with the given plane.

10. Through a line inclined at 30° draw two planes inclined at 40° and 45° respectively. How many solutions does this problem admit?

11. Two indefinite lines contain an angle of 60°, one is inclined at 50°, the other at 40°: draw their plan and determine the inclination of the plane containing them.

12. The horizontal trace of a plane makes an angle of 60°, and the vertical trace an angle of 50°, with xy: draw a straight line parallel to this plane and inclined at 20°, passing through a point 2 in. high and 2 in. distant from the plane.

13. Through a point 3 in. high in the V. P. draw a line making an angle of 30° with that plane and of 45° with the H. P.

14. From a point 2 in. high in a plane inclined at 50°, determine (α) a perpendicular to the plane, (β) a straight line lying in the plane and inclined at 25°. Find the inclination of the plane containing the lines (α) and (β).

15. From a point 3 in. high draw indefinite straight lines AB, AC at right angles to each other, AB inclined at 30° and AC at 20°.

16. Draw the plan of a right-angled triangle ABC lying in a plane inclined at 45°: the side AB being inclined at 25. AB =BC=2·5 in.

17. Draw the plan of the regular pentagon, side 2·5 in., when two of its adjacent sides are inclined at 15° and 30° respectively.

18. Draw the traces of a plane inclined at 50° and perpendicular to the V. P. Determine the angle which a horizontal line making an angle of 50° with $x\ y$ makes with the inclined plane.

19. Two lines, inclined, one at 20°, the other at 40°, lie in a plane inclined at 50°. Find the angle between them. Solve both cases.

20. Two lines meeting in a point make an angle of 100°. The angle between the lines of which these are the plans is 90°: they are equally inclined to the horizon: find the inclination of the plane containing them.

21. A plane is inclined at 50° to the H. P. Its horizontal trace makes an angle of 30° with $x\ y$: determine, by its traces, a second plane perpendicular to the first, and inclined at 60° to the H. P.

22. Two of three planes mutually perpendicular are inclined at angles 40° and 65° to the H. P.; at what angle is the third plane inclined?

23. Draw the plan of a line, 3 in. long, inclined at 60°, and the plan of the locus of the extremities of lines 2 in. long drawn from the upper end of the first line making angles of 30° with it, downwards.

24. The three plane angles of a trihedral angle are 35°, 40°, 50°: determine its dihedral angles.

25. The three dihedral angles of a trihedral angle are 145°, 140°, 130°: find its plane angles.

26. Two sides of a spherical triangle are 84° and 44°, the included angle is 36°: find the remaining parts.

27. Two sides of a spherical triangle are 80° and 109°, the angle opposite the first is 33°: find the other parts.

28. In a spherical triangle A B C, given $A = 47°$, $B = 52°$, $a = 56°$: solve the triangle.

29. The horizontal trace of a plane makes an angle of 30° with $x\ y$ and the vertical trace an angle of 50°, find the point

in which a line parallel to $x\,y$, 2 in. above the H. P. and 1·5 in. from the V. P., meets this plane.

30. The angle observed from a point A to two points B and C, of which the altitudes above the horizon from the same point are 30° and 35° respectively, is 45°: construct the plan of the angle between the lines A B and A C.

CHAPTER III.

HORIZONTAL PROJECTION.—INTRODUCTION.

1. It has been shown in the preceding chapters that all points and lines in space can be represented by means of their projections on two planes cutting each other at right angles. Sometimes, however, it is more convenient to represent objects by their plans alone. This is especially the case in fortification and topographical drawings, in which, since many of the lines are nearly horizontal, the elevations would often intersect beyond the limits of the drawing. The elevations thus dispensed with are replaced by indices, affixed to the plans of the various points, denoting the vertical distances of those points from a given horizontal plane called the *plane of reference or comparison.* This kind of projection is known as *horizontal projection. A figured plan* is a plan which has attached to it a number, showing, in units of the scale of the drawing, the vertical distance from the plane of reference of the point of which it is the plan. This number, as stated above, is termed the *index of the point.* The indices of all points in the plane of reference will evidently be zero; whilst those of all points below that plane will be negative. Negative indices may however be avoided by assuming the plane of reference to have an elevation not greater than that of the lowest point in the drawing. In the following problems the horizontal plane of projection will be assumed as the plane of reference. The horizontal trace of a line will then be that point in the line whose index is zero. This will be termed the zero point of the line. Again, since all points in a horizontal line, or in a horizontal plane, are equidistant from the plane of reference, they must evidently have the same index; and when such line is the horizontal trace of a plane, its index will be zero, and it will be simply called the trace of the plane.

2. As an illustration of the foregoing definitions it may be stated that the point lettered thus, P, p, denotes a point in space whose plan is P, and whose vertical distance from the plane of reference is p units of the scale of the drawing. If the unit were ·1 inch, $P, 3$ would be a point ·3 in. above the plane; $P, -3$ a point ·3 in. below it.

It is manifest that a point is known when given by its figured plan. For any straight line drawn in the plane of reference may be considered as the intersection of that plane by a vertical plane; and on this plane an elevation may be made in accordance with Chap. I. This is what must be understood when in the following problems the expression 'make an elevation of the point' occurs.

A straight line is determined by the figured plans of two of its points, since only one straight line can pass through the same two points.

The line mentioned above corresponds to the axis or ground line in Chapters I. and II.; it will, however, in this Chapter, be termed the *line of level*; and, unless the contrary be stated, its index will always be zero.

3. When two straight lines intersect in space, their plans also cut each other, and the point of intersection has the same index in both plans. It can therefore be ascertained whether two lines, whose plans have a common point, cut each other or not.

4. Since parallel straight lines are equally inclined to the plane of reference, their plans, which will be parallel, will have their indices increasing uniformly in the same direction. In other words, equidistant points on the plans of two parallel lines will have equidifferent indices. This affords a means of determining whether lines given by their plans are parallel or not.

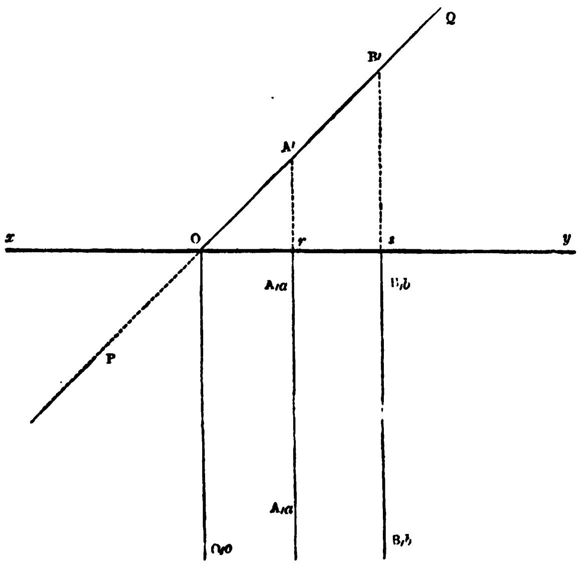
5. *The contours* of a surface are the lines in which it is intersected by a series of equidistant horizontal planes. From this definition it follows that the contours of a plane are a series of equidistant horizontal straight lines, parallel to the horizontal trace of the plane.

6. *The horizontals* of a plane are the plans of its contours, and

are therefore a series of equidistant straight lines parallel to the horizontal trace of the plane.

A plane is adequately represented by two of its figured horizontals. This is evident; for if a line of level (2) be drawn at right angles to the horizontals, it will be the horizontal trace of a vertical plane at right angles to the given plane, and to its contours. The traces of the contours on this plane may be at once determined, since they will be situated in the horizontals produced, and at distances from the line of level equal to their respective indices. These traces will be two points in the trace

Fig. 45*a*.



of the given plane; the straight line joining them will be that trace, and if produced to meet the line of level, the angle which it makes with that line will be the angle at which the plane is inclined to the horizontal plane. The straight line drawn through the point in which the trace and line of level intersect, parallel to the horizontals, will be the horizontal trace of the plane.

To illustrate this, let A,*a* and B,*b* (Fig. 45*a*) be two horizontals

of a plane. Draw $x\,y$ at right angles to these horizontals, and meeting them in r and s respectively, make r A′ equal to a, s B′ equal to b; then the straight line P Q passing through A′ and B′, and meeting $x\,y$ in O, will be the trace of the given plane; the angle Q O y will be its inclination to the horizontal plane; and the straight line O,o, drawn through O, parallel to A,a and B,b, will be the horizontal trace of the plane.

This construction will sometimes be called 'making an elevation of the given plane on the line of level $x\,y$,' because the line P Q contains the elevations of all points and lines in the given plane, when the vertical plane, whose trace is $x\,y$, is considered as the plane of projection.

7. The horizontals of parallel planes will be parallel (*Euc.* XI. 16), and their indices will increase uniformly in the same direction. That is, equidistant horizontals in the two planes will have equidifferent indices. This property affords a means of ascertaining whether planes are parallel or not.

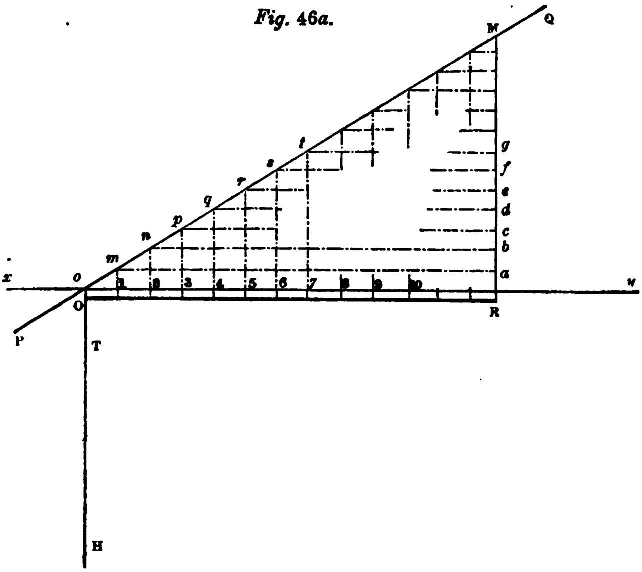
8. *The scale of a plane* is a straight line, perpendicular to its horizontal trace, graduated at the points in which it would be cut by equidistant horizontals of the plane, the points of graduation having attached to them the indices of the corresponding horizontals.

Let H T (Fig. 46*a*) be the horizontal trace of a given plane. Draw $x\,y$ at right angles to H T, and meeting it in O. Through O draw P Q, making with $x\,y$ an angle equal to the inclination of the given plane to the horizontal plane. From any point M in P Q draw M R perpendicular to $x\,y$; set off, on R M, R $a = a\,b = b\,c = c\,d =$ &c. &c. &c. $=$ 1 unit of the scale of the drawing; draw $a\,m$, $b\,n$, $c\,p$, $d\,q$, &c. &c. &c. parallel to $x\,y$, and meeting P Q in m, n, p, q, &c. respectively; draw through m, n, p, q, &c. straight lines parallel to M R, and meeting $x\,y$ in the points marked 1, 2, 3, 4, &c.; $x\,y$ will be the scale of the plane graduated according to the definition. This will be evident if the profile plane Q O R be supposed to take a vertical position.

9. The scales of parallel planes will be parallel, since they are perpendicular to the horizontals which are parallel (7).

If P Q be the elevation of a straight line whose plan is $x\ y$; the graduated scale O R will evidently be the scale of the line. In order to distinguish between the scale of a plane and that of a

Fig. 46a.



line, it is usual in practice to draw the scale of a plane as shown in the figure; whereas the scale of a line is represented by a single graduated straight line only.

10. When the term 'inclination' is used, without any qualification, it must be understood to signify the inclination of a line, or a plane, to the plane of reference.

Problem I.

Through a given point to draw a straight line having a given inclination.

Let A, a (Fig. 47) be the given point. Through A, in the horizontal plane, draw a straight line $x\ y$: on $x\ y$ as a line of

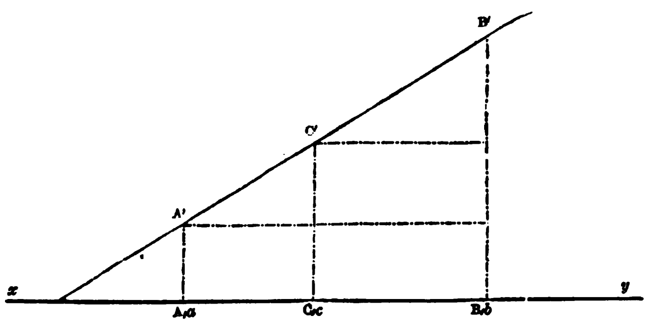
level, make an elevation of A by drawing A A′ perpendicular to $x\ y$, and making A A′ equal to a. At the point A′ in A A′ make the angle A A′ O equal to the complement of the given inclination. Let A′ O cut $x\ y$ in O: the angle A O y being equal to the given angle, A′ O is the line required. O is a point in its plan having an index zero: the indices of two points in the line being thus known, the line is determined if the position of $x\ y$ is known; otherwise the problem admits of an infinite number of solutions; since all straight lines passing through A′, and meeting the horizontal plane at a distance from A equal to A O, will have the same inclination. (Ch. I. 33, *Cor.* I.)

Problem II.

Given the plan of a line; to determine (1) its inclination; (2) the length of the line between two points whose indices are given; (3) a point in the plan having a given index; (4) the index of a given point.

1. Let A,a and b,B (Fig. 49) be the two given points in the plan $x\ y$. Assuming $x\ y$ as a line of level, draw A A′ equal to a,

Fig. 47.



and B B′ equal to b, perpendicular to $x\ y$. Join B′ A′ and produce it to meet $x\ y$ in O: the angle B O y will be the inclination required. (Chap. I. 33.)

2. The line, of which A B is the plan, is the hypothenuse of a right-angled triangle, whose base is equal to A B; and whose perpendicular is $a \sim b$; A′ B′ is therefore the line required. If a and b had opposite signs, A A′ and B B′ would have been drawn on opposite sides of $x\ y$. The perpendicular would then have been $a + b$. This construction applies to (1) also.

3. Let C,c be any point in A B; draw C C′ perpendicular to $x\ y$; through A′, B′, C′ draw straight lines parallel to $x\ y$: then by similar triangles,

$$\begin{aligned} a \sim b : a \sim c &:: \mathrm{A'B'} : \mathrm{A'C'}. \\ &:: \mathrm{A\,B} : \mathrm{A\,C} \quad (\alpha); \end{aligned}$$

therefore, $\mathrm{A\,C} = \frac{a \sim c}{a \sim b} \times \mathrm{A\,B}$, which determines A C, since a, b, c and A B are known.

4 $\mathrm{A\,B} : \mathrm{A\,C} :: a \sim b : a \sim c \quad (\beta)$;

therefore, $\sim \ (\ a \quad b\ ) \times \frac{\mathrm{A\,C}}{\mathrm{A\,B}}$, which added to the index of A gives the index of C.

This construction has been based on the assumption, that the index of the horizontal line $x\ y$ is zero. Should this not be the case, the index of that line will have some definite value as $\pm\ d$, which must in every case be added, with its proper sign, to the index obtained.

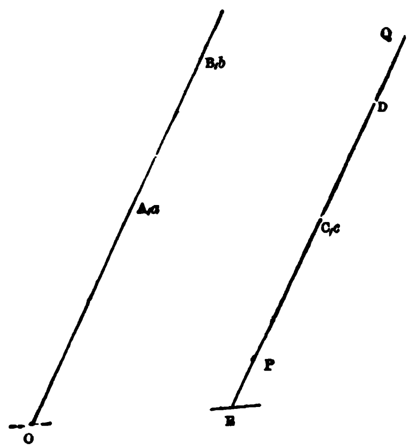
N.B. The readiest way of finding A C in (α) and $a \sim c$ in (β) will be to use the line of lines on the sector.

Problem III.

Through a given point to draw a straight line parallel to a given straight line.

Let A,a and B,b (Fig. 48) be two points in the plan of the given line: C,c the given point. Then since the lines are to be parallel their plans will be parallel. (Chap. I. 11.) Draw through C a straight line P Q parallel to A B: P Q will be the plan of the required line. Make C D equal to A B; D,d will be a point whose index $d = \mathrm{C} \pm (a \sim b)$ according as b 7 or $\angle$ a; and the line is determined.

Fig. 48.

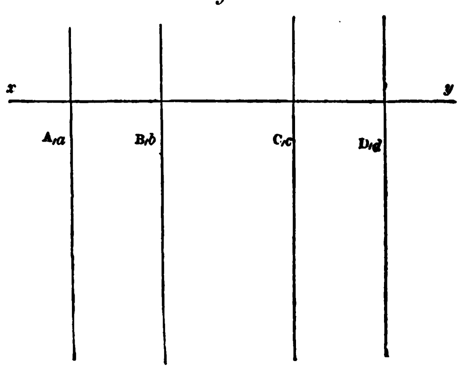


PROBLEM IV.

Through a given point to draw a plane parallel to a given plane.

Since the planes are to be parallel, their horizontals will be parallel. (*Euc.* XI. 16.)

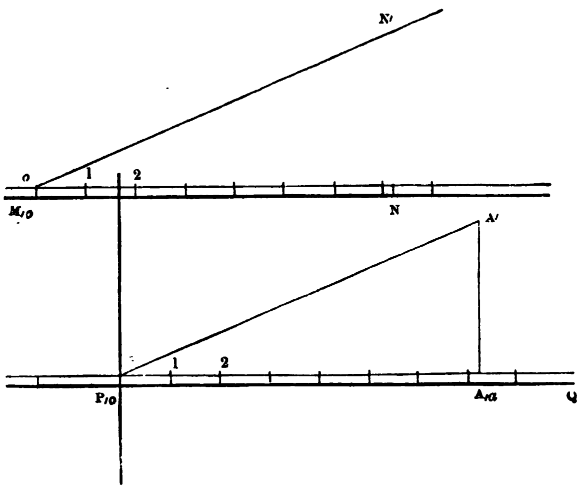
Fig. 49.



Let $A_{/}a$ and $B_{/}b$ (Fig. 49) be two horizontals of the given plane: $C_{/}c$ the given point. Draw $x\ y$, as a line of level, perpendicular to the horizontals. The line $C_{/}c$ perpendicular to $x\ y$, will be a horizontal of the required plane. Make C D equal to A B: through D draw a straight line parallel to $C_{/}c$: this will be another horizontal of the required plane; and its index d will evidently be equal to $c - (a \sim b)$, according as b is greater or less than a.

Otherwise, let M N (Fig. 52) be the scale of the given plane, the index of M being zero: $A_{/}a$ the given point. Through A

Fig. 50.



draw P Q the scale of the required plane, parallel to M N. Let N be the point whose index is a: make A P equal to M N: P will be the zero point of the scale of the plane required which can at once be graduated from that of the given one.

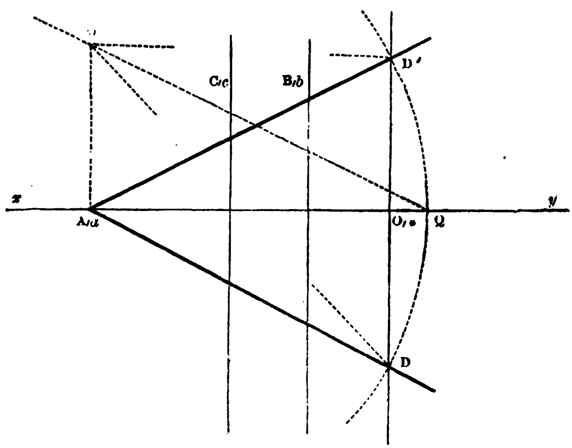
Problem V.

Through a given point in a given plane, to draw a straight line that shall have a given inclination.

The inclination of the line cannot be greater than that of the plane. (Chap. I. 33.)

If the inclination of the line be equal to that of the plane, only one line can be drawn. In this case the problem will be solved by drawing through the given point a straight line, per-

Fig. 51.



pendicular to the horizontals of the plane: this will be the plan of the required line: and the points in which it cuts the horizontals will have the same indices as the horizontals.

If the inclination of the line be less than that of the plane, the problem will be indeterminate, since two lines can be drawn, lying in the given plane, and making the given angle with the horizontal plane.

Through A,a, the given point (Fig. 51), draw $x\ y$ perpendicular to C,c and B,b, the horizontals of the given plane. On $x\ y$, as a

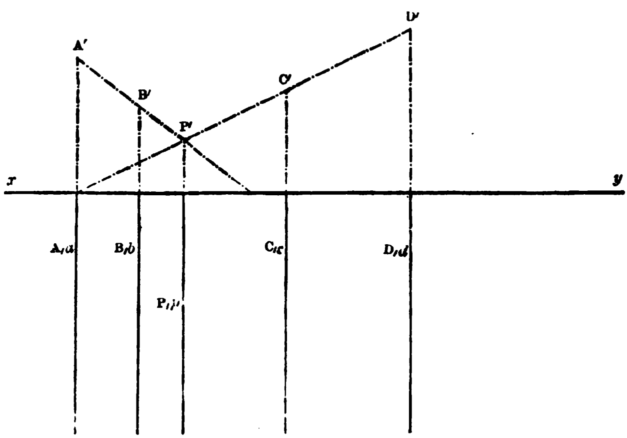
line of level, make an elevation A′ of the given point $A_{/}a$: find the point in $x\,y$, whose index is zero (Prob. II.), let it be O: through O draw O D, parallel to the given horizontals: this will be the trace of the given plane. At the point A′ in A A′ make an angle A A′ Q, equal to the complement of the proposed inclination. Let A′ Q cut $x\,y$ in Q; A′ Q will be a line having the proposed inclination: A Q its plan. With A as a centre and radius A Q, describe a circle cutting D D′ in D and D′: join A D and A D′; these lines will be the plans (Chap. I. 33) of the two lines which fulfil the conditions.

Problem VI.

To find the intersection of two given planes.

1. Let the horizontals $A_{/}a$, $B_{/}b$ and $C_{/}c$, $D_{/}d$ (Fig. 52) of the given planes be parallel. Draw $x\,y$ perpendicular to the hori-

Fig. 52.

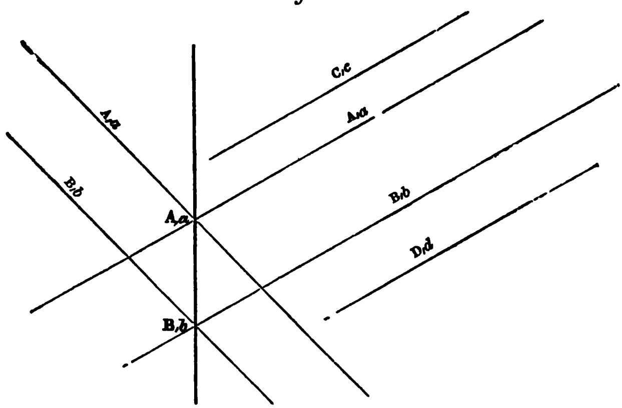


zontals; on $x\,y$, as a line of level, make elevations A′ B′, C′ D′ of the planes. Through P′ the point in which these elevations intersect, draw $P_{/}p$, parallel to $A_{/}a$ or $C_{/}c$; $P_{/}p$ will be the plan

of the intersection of the planes, and its index p may be readily determined from the indices of the given horizontals. (Prob. II.)

2. If the horizontals of the two planes be not parallel, let $A_{,}a$ and $B_{,}b$ (Fig. 53) be the horizontals of one plane; $C_{,}c$ and $D_{,}d$ those of the other: in this second plane find (Prob. II.) two horizontals $A_{,}a$ and $B_{,}b$: the points in which these respectively intersect the corresponding horizontals of the first plane, will be two points in the plan of the required intersection. The straight line drawn through these points will be the plan required: and,

Fig. 53.

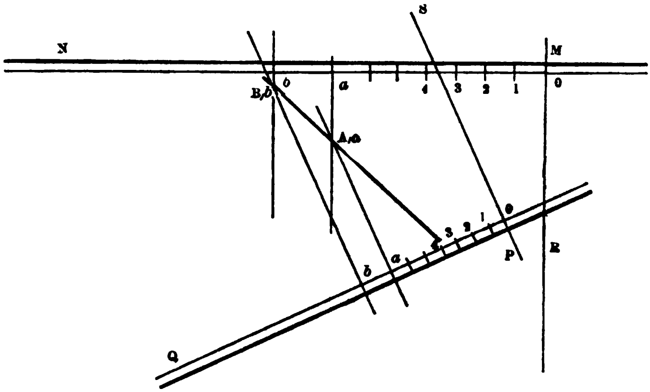


since the indices of two points in it are known, it is fully determined.

Otherwise, let M N and P Q (Fig. 54) be the scales of the given planes; M R and P S being their traces: through the points of the scales figured a and b in each draw straight lines parallel to the traces of the respective planes. Let these parallels intersect in $A_{,}a$ and $B_{,}b$: the indefinite straight line passing through A and B will evidently be the plan of the intersection.

The solutions of the cases when the scales are parallel, and when one of the planes is perpendicular to the plane of reference, may be drawn as an exercise.

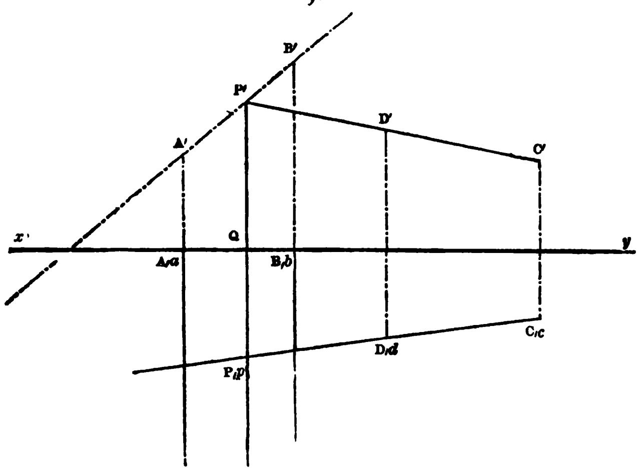
Fig. 54.



PROBLEM VII.

To find the intersection of a given straight line and a given plane.

Fig. 55.



Let A,*a* and B,*b* (Fig. 55) be the horizontals of the given plane: D C the plan of the line. Draw *x y* perpendicular to

A, a or B, b: on $x\,y$ as a line of level, make elevations of the plane and the line.

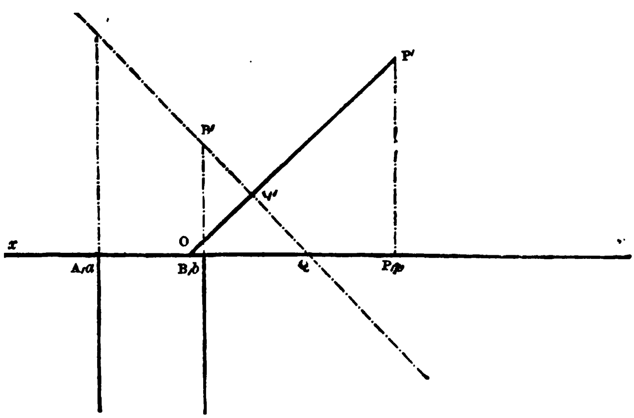
The point P′ in which C′ D′, the elevation of the line, cuts A′ B′, the elevation of the plane, will be the point required. For the elevation of the point in which the line meets the plane must be in A′ B′ and in C′ D′; it will, therefore, be P′, the point in which these lines intersect. Let P′ P drawn parallel to D D′, cut C D in P: P will be the plan of the point of intersection: its index p being equal to P′ Q.

Problem VIII.

To draw a straight line through a given point perpendicular to a given plane.

Let A, a, B, b (Fig. 56) be two horizontals of the given plane; P, p the point. Since the line is to be perpendicular to the

Fig. 56.



plane, its plan will be perpendicular to the trace of the plane. (Chap. I. 31.)

Let $x\,y$ be this plan; on it make an elevation A′ B′ of the plane; and an elevation P′ of the point. Then the elevation of

the line will be perpendicular to that of the plane (Chap. I. 31). Draw P′ Q′ perpendicular to A′ B′; this will be the elevation of the line; and since P′ Q′ lies in the vertical plane passing through $x\ y$, it is evident that P′ Q′ measures the distance of the given point from the given plane. If P′Q′ meet $x\ y$ in O: P′ O P will be the inclination of the line.

Problem IX.

Through a given point to draw a plane perpendicular to a given line.

Let $x\,y$ (Fig. 56) be the plan of the line, P,p a point in it, B,b the given point. P′ O being the elevation of the line, B′ that of the given point; through B′ draw A′ B′ perpendicular to P′ O; A′ B′ will be the elevation of the plane: and since the indices of the horizontals passing through B and Q are known, the plane is determined, and may be easily represented either by its horizontals or by its scale.

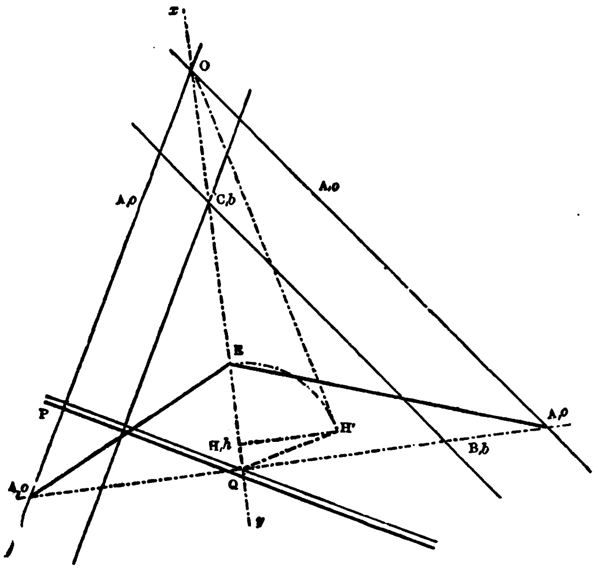
Problem X.

To determine the angle between two given planes.

Determine the traces A,o of each plane and two horizontals B,b: let these horizontals intersect in C, and the traces in O (Fig. 57); the straight line $x\ y$ drawn through O and C will be the plan of the intersection of the planes (Prob. VI.) In $x\,y$ find apoint H,h; draw H H′ perpendicular to $x\ y$, and h units in length. Join O H′; from H′, draw H′ Q perpendicular to O H′ and meeting $x\ y$ in Q: through Q draw A Q A″ perpendicular to $x\ y$: meeting the traces in A and A″. Make Q R equal to Q H′ join A R and A″R: the angle A R A″ will be the angle required (Chap. I. 36).

This problem may be solved in a similar way when the planes are given by their scales.

Fig. 57.



PROBLEM XI.

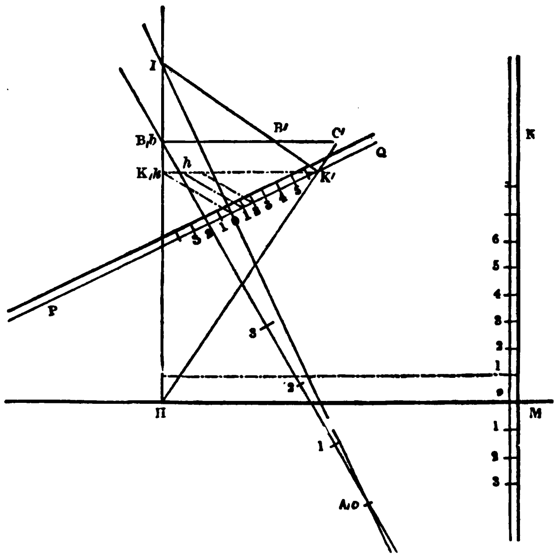
Through a given straight line to draw a plane perpendicular to a given plane.

If from any point in the given line a straight line be drawn perpendicular to the given plane. The plane containing this perpendicular and the given line will be the plane required.

Let M N (Fig. 58) be the scale of the given plane; H M being its trace; A B the plan of the given line; the index of A being zero, that of B, *b*. Through B draw H I perpendicular to H M: B C′ parallel to H M: make B B′ equal to *b*; and B C′ equal to the index of the corresponding point of the plane. Join C′ H, and draw K′ I through B′ perpendicular to C′ H, and meeting H I in I; I will be the trace of this perpendicular (Prob. VIII.) The trace of the required plane must pass through I and also through A, the trace of the given line. I A is there-

fore the trace required. Draw K′ K parallel to HM; KK′ will be equal to the index of a point in the plane. If the scale line P Q be drawn perpendicular to the trace I A, and graduated, the problem is solved.

Fig. 58.



To graduate the scale: let I A and P Q intersect in 0: join K 0: in K K′ make K *h* equal to one unit of altitude: draw *h* 1 parallel to K 0 and meeting P Q in 1; 0 1 will be the first division of the scale; which may therefore be completed by setting off along P Q distances equal to 0 1, and affixing the proper indices.

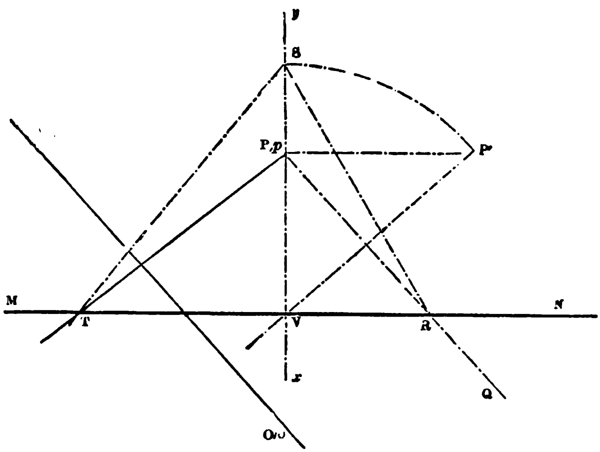
Problem XII.

Through a given point in a given plane, to draw a straight line, which shall make a given angle with a given straight line in that plane.

Let M N (Fig. 59) be the trace of the plane, A O the figured plan of the given line; P,*p* the given point: through P draw

P Q parallel to A O, meeting M N in R, and $x\,y$ perpendicular to M N; draw P P′ perpendicular to $x\,y$, and equal to p units of altitude. Join P′ with V the point in which $x\,y$ cuts M N; make V S equal to V P′: join R S; make the angle R S T equal

Fig. 59.



to the given angle. Join T P; T P will be the plan of the required line, which may be figured by drawing parallels to M N.

This construction is obvious, since the triangle T P R is evidently the plan of a triangle identical with T S R. (Chap. I. 35.)

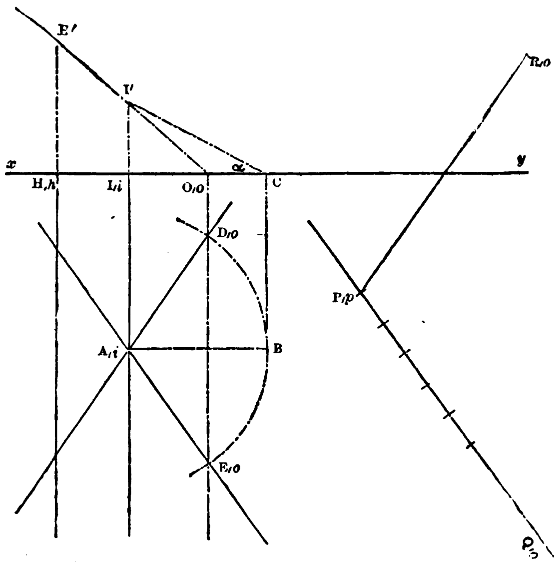
Problem XIII.

Through a given point to draw a straight line, having a given inclination and parallel to a given plane.

Let H,h, I,i (Fig. 60), be two horizontals of the given plane, P,p, the given point: on $x\,y$ as a line of level at right angles to the horizontals, make an elevation H′ I′ of the plane, and let

H′ I′ meet x y in O: through O draw O E perpendicular to x y; O E will be the trace of the plane. At I′ make the angle I I′ C, equal to the complement of the given inclination. From any point A in the horizontal $I_{/}i$, draw A B perpendicular to $I_{/}i$, and equal to I C: with A as a centre and radius A B, describe a circle cutting the trace O E in D and E: D and E will be the

Fig. 60.



zero points of two lines in the plane, having the given inclination: the plans of these lines will evidently be A D and A E. Through $P_{/}p$ draw P Q and P R parallel to A E and A D respectively; P Q and P R will be the plans of two lines fulfilling the conditions of the problem. The index of P and the inclination of the lines being given, the plans may readily be figured.

N.B.—The inclination of the line must not be greater than that of the plane. (Chap. I. 33.) The problem, depending upon the intersection of a straight line and a circle, will evidently admit either of two solutions, or of one solution, or be impossible, accordingly as the circle cuts O D, touches

O D, or does not meet O D: the third case arises when the inclination of the line is greater than that of the plane; the second when it is equal to it; the first, when it is less.

PROBLEM XIV.

To draw a plane, making a given angle with a given plane, and containing a given line in that plane.

The line will evidently be the intersection of the given and required planes.

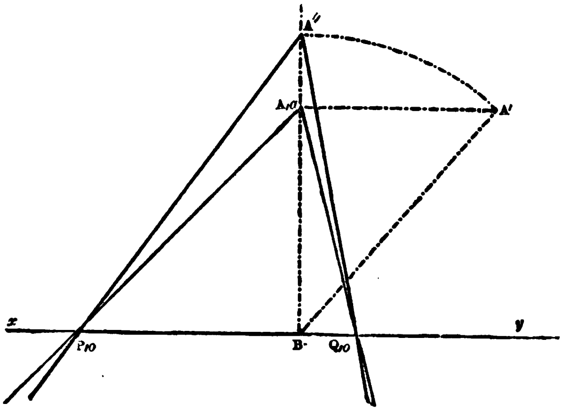
Let $A_{/}o$ (Fig. 57) be the trace of the given plane, P Q its scale, x y the plan of the given line; in x y, take a point $H_{/}h$, draw H H′ perpendicular to x y and equal to h units of altitude; join H′ with O the trace of the line; draw H′Q perpendicular to O H′ and meeting x y in Q; through Q draw A Q A″ perpendicular to x y; make Q R equal to Q H; join A R; draw R A″, making with A R an angle equal to the profile angle of the two planes, and cutting A A″ in A″; A″ will be a point in the trace of the required plane, and O is another point in the trace, consequently the straight line A″ O is the trace; and since $H_{/}h$ is a point in the plane, if the scale be drawn through H perpendicular to the trace, it can be graduated at once.

PROBLEM XV.

Given the plans of two lines, to determine the angle contained by the lines.

Let A P and A Q (Fig. 61) be the plans of the lines, P and Q being their traces, $A_{/}a$ the point of intersection; then the straight line P Q will be the trace of the plane containing the lines. Through A draw A B perpendicular to P Q; draw A A′ perpendicular to B A, and equal to a units of altitude; in B A produced make B A″ equal to B A′, join P A″ and Q A″; the angle P A″ Q will be the augle required. (Chap. I. 36.)

Fig. 61.

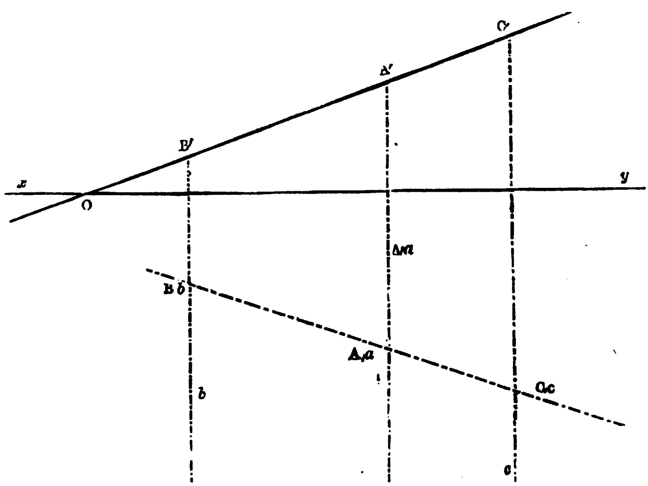


PROBLEM XVI.

To determine the horizontals and the inclination of a plane containing three given points.

Let A,a, B,b, C,c (Fig. 62) be the given points.

Fig. 62.



Join any two of these points as B and C; in the line B C, find a point whose index is a (Prob. II.). Join this point with A,a: the line thus drawn will be a horizontal of the required plane, having an index a; a line drawn through B or C parallel to A,a will be a second horizontal of known index, and the plane is therefore determined.

Draw $x\ y$ at right angles to these horizontals; on it as a line of level make elevations A', B', C' of A, B, C respectively. A', B', C', being points in the vertical trace of a plane, should be in a straight line. The angle $C'\ O\ y$ which this line makes with $x\ y$ is evidently the inclination of the plane.

Problem XVII.

To find the angle between a given straight line and a given plane.

Let P Q (Fig. 63) be the trace of the given plane, a its inclination, A B the plan of the line.

Draw $x\ y$ at right angles to P Q, on it as a line of level make elevations, Q R of the plane, and $a'\ b'$ of the line; through b' the trace of the line, draw $m\ n$ parallel to $x\ y$: make $m\ n$ equal to A B, and $a'\ o$ equal to $a'\ n$: the angle $a'\ o\ b'$ will be the angle required.

Problem XVIII.

To draw a straight line perpendicular to two given straight lines.

Let A F, C D (Fig. 64) be the plans of the lines; through any point C,c of the lower line draw a line parallel to the upper line, find its trace E, o; and the trace H, o of the lower line; join H E: this will be the trace of a plane, containing the lower line and parallel to the upper; from Q in H E produced draw $x\ y$ perpendicular to H Q; on it make an elevation of the plane, and from A, a draw a perpendicular to it: through the foot of this perpendicular draw a line parallel to the upper line, and in the plane, to

Fig. 63.

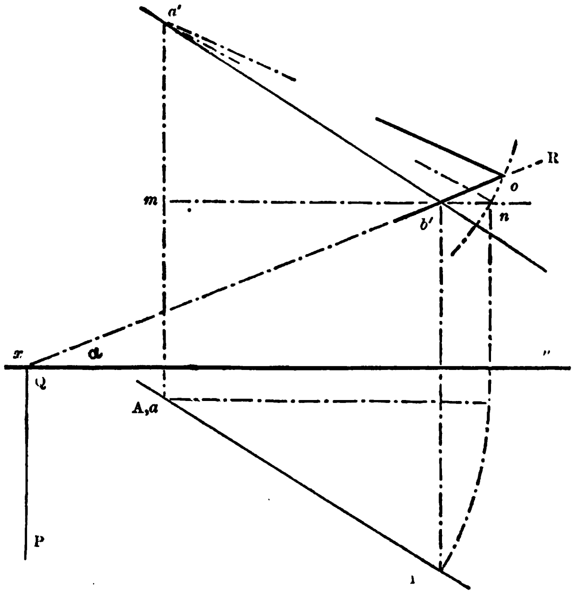
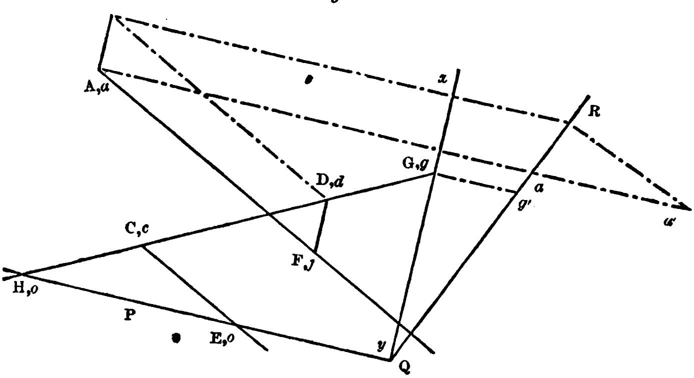


Fig. 64.

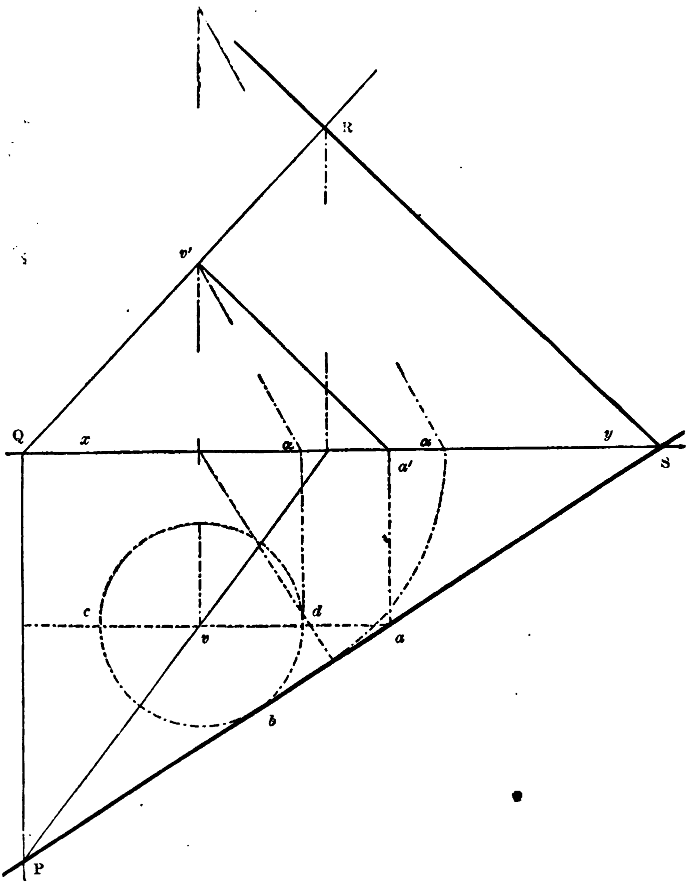


meet the lower line; from the point of intersection draw a line parallel to the perpendicular, this will be the perpendicular sought.

Problem XIX.

To draw a plane perpendicular to a given plane and inclined at a given angle.

Fig. 65.



Let P Q R (Fig. 65) be the given plane, its horizontal trace being perpendicular to $x\ y$, α being the given angle. Make any

point (v, v') in the plane P Q R the vertex of right cone whose generatrix makes an angle α with the horizontal plane on which it stands, b c d being its base. From (v, v') draw a straight line (v a, v' a') perpendicular to the given plane; from its trace a draw a tangent to the circle b c d: this will be the horizontal trace of the required plane; its vertical trace S R can readily be determined since the plane contains the vertex of the cone.

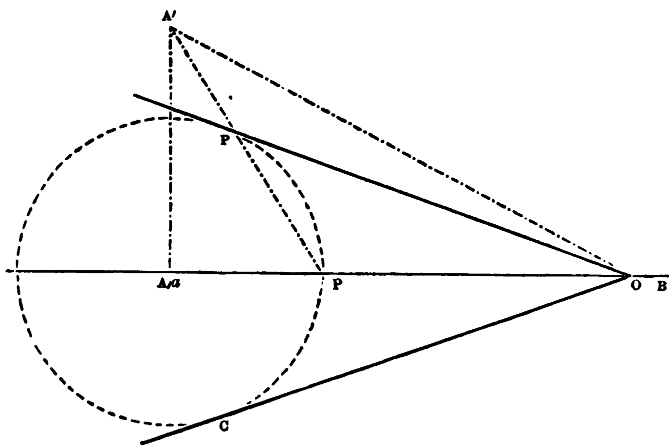
Problem XX.

To draw a plane to contain a given line and have a given inclination.

1. If the line be horizontal, it will be a horizontal of an indefinite number of planes.

2. If the line be not horizontal, let A B (Fig. 66) be its plan, on A B, as a line of level, make an elevation A′ of A.

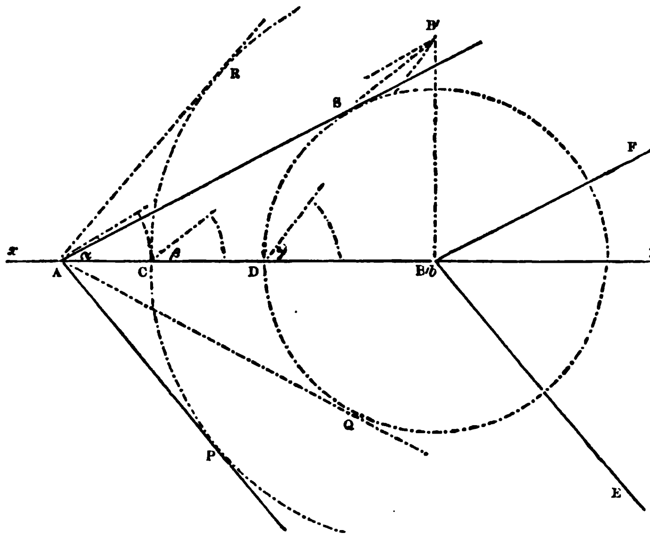
Fig. 66.



At A′ in A A′, make the angle A A′ P equal to the complement of the given angle of inclination. Let A′ P cut A B in P; with A as a centre, and radius A P, describe a circle C P D; from

O, the zero point of A B, draw O C and O D, touching this circle in C and D. O C and O D will be the horizontals of two planes fulfilling the proposed conditions. The inclination of the plane

Fig. 67.



being given, other horizontals may be determined, having given indices.

NOTE.—The inclination of the plane must not be less than that of the line. (Chap. I. 33.) If the inclination of the plane be equal to that of the line, there will be only one solution, P and O will coincide, and the horizontal trace of the plane will pass through O and be perpendicular to A B.

PROBLEM XXI.

Through a straight line inclined at a given angle, to draw two planes inclined at given angles.

Let the angle of inclination of the line be α; those of the planes being β and γ respectively.

Assume the line to be projected in $x\ y$ (Fig. 67) at any point A in $x\ y$; make the angle B A B′ equal to α, and from a second

point B draw B B′ perpendicular to $x\ y$; A will be the trace of the line; B, b (if B B′ be equal to b) will be a point in both planes.

Make the angle B C B′ equal to β, and B D B′ equal to γ: B′ b may be assumed as the common vertex of two right cones whose generatrices are inclined at angles β and γ respectively to the horizon: the required planes must be tangents to these cones, each to each; their traces must pass through A, and touch the traces of the cones (Chap. I. 43).

The following will therefore be the construction:—

With B as a centre and radii B C, B D describe the circles P C R, Q D S: from A draw A P and A S touching these circles in P and S: A P and A S will be the traces of two planes fulfilling the conditions. Through B draw B E parallel to A P; B F parallel to A S; these lines will be horizontals of the planes having an index b, the planes are therefore determined.

This problem admits of four solutions, as shown in the figure.

Cor. If the angles α, β (or γ) and B A S, were given, the angle γ (or β) might readily be found, as is evident from the construction.

Problem XXII.

To draw a tangent plane to a given cone through a given point on its surface.

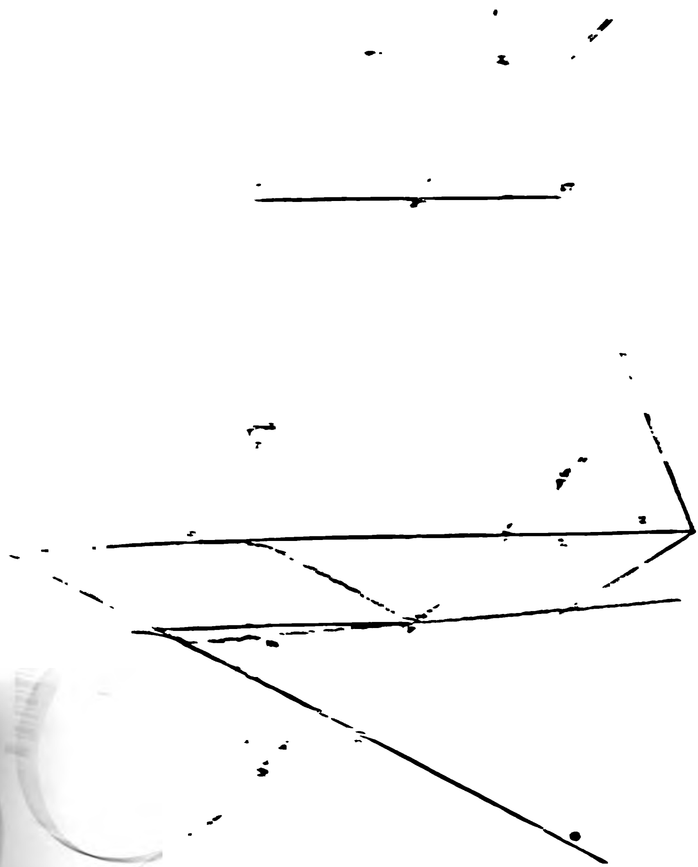
Let $a\ c\ b\ d$ (Fig. 68) be the horizontal trace of the cone; (v, v') its vertex.

To determine the limits within which the projections of all generatrices must lie: draw in the horizontal plane the tangents $v\ a$, $v\ b$; the plans of all generatrices will lie between $v\ a$ and $v\ b$. Draw $c\ c'$ and $d\ d'$ tangents perpendicular to $x\ y$; join $v'\ c'$ and $v'\ d'$. The elevations of all generatrices will be between $v'\ c'$ and $v'\ d'$.

Next, to find the points on the surface corresponding to a given projection: let m' be the given projection. Draw the straight line $v'\ m'$, cutting $x\ y$ in e'; draw $e'\ e$ perpendicular to $x\ y$; $v'\ e'$

[illegible]

[illegible] perpendicular [illegible] the vertical [illegible] and passing [illegible] the surface of [illegible]



the cone in two generatrices [illegible] and since its horizontal trace meets [illegible] it is manifest that the plans of the generatrices are the straight lines [illegible] and [illegible] Draw m'm perpendicular to [illegible], meeting [illegible] in m and [illegible] in n: m and n will be the plans of the points whose elevation is m'.

To draw a tangent plane through the point (s, s'), it must be remembered that this plane will contain the generatrix ($v f$, $v' f'$), and touch the cone throughout the entire extent of that line (Chap. I.) Draw P S, touching $a\,c\,b\,d$ in f; find (Chap. II. Prob. III.) t' the vertical trace of ($v f$, $v' f'$); through t' draw RS: P S R will be the plane required.

In the same way a second tangent plane may be determined, as P Q T, containing the generatrix ($v\,e$, $v' e'$).

If through any point in the generatrix a straight line ($v r$, $v' r'$) be drawn parallel to the trace P Q, the vertical trace of this line will evidently be a point in the vertical trace of the plane. This sometimes affords a means of determining the vertical trace of the plane, when that of the generatrix falls beyond the limits of the drawing. For example, the vertical trace Q T was found by means of r', the trace of ($v\,r$, $v' r'$), because that of ($v\,e$, $v'\,e'$) could not be conveniently determined.

Problem XXIII.

To draw a tangent plane to a given cone through a given point without it.

Every plane touching the cone passes through the vertex (Chap. I. 43), and has its horizontal trace a tangent to the base of the cone.

The problem may therefore be solved as follows;—

Join the given point with the vertex of the cone; determine the traces of the line joining these points (Chap II. Prob. III.); through its horizontal trace draw tangents to the base of the cone. These tangents will be the horizontal traces of two planes, fulfilling the conditions of the problem. Their vertical traces will be the straight lines drawn through the vertical trace of the said line, and the points in which the horizontal traces meet the ground line.

PROBLEM XXIV.

To draw a tangent plane to a given cone parallel to a given straight line.

Since the plane must pass through the vertex of the cone, the problem may be solved by the following construction :—

Through the vertex of the cone draw a straight line parallel to the given line. (Chap. II. Prob. II.) Determine the traces of this line. (Chap. II. Prob. III.) Through its horizontal trace draw a tangent to the base of the cone. This will be the horizontal trace of the required plane. Through the point in which this trace meets the ground line, and the vertical trace of the line passing through the vertex, draw a straight line; this will be the vertical trace of the plane.

PROBLEM XXV.

To draw a straight line through a given point to make a given angle with one given plane and to be parallel to another given plane.

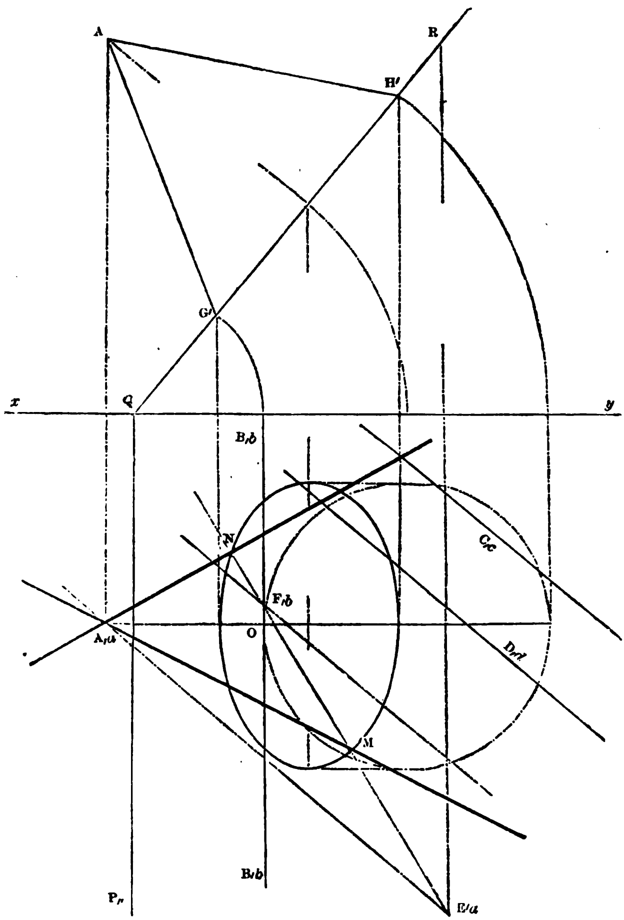
The angle between the line and the plane must not be greater than that contained by the two planes. (Chap. I. 33.)

Let $A_{/}a$ (Fig. 69) be the given point; P Q the trace, $B_{/}b$ a horizontal of the first plane; $C_{/}c$ and $D_{/}d$ two horizontals of the second plane. Draw $x\,y$ perpendicular to P Q, and on $x\,y$ as a line of level make an elevation Q R of the first plane: assume $A_{/}a$ to be the vertex of a right cone, with circular base, standing on the plane P Q R; so that A′ G′ H′ is its elevation; the curve M O N being the plan of the base of the cone (Chap. II. Prob. XL.); and the angle A′ G′ H′ equal to the given angle.

Through $A_{/}a$ draw a plane parallel to the second (Prob. IV.): find (Prob. VI.) E F the intersection of this plane with the plane P Q R: let E F meet the plan of the base of the cone in M and N: through A draw A M and A N: these will be the plans of

two lines fulfilling the conditions of the problem. For, being the plans of the generatrix of the cone in two positions, they make

Fig. 69



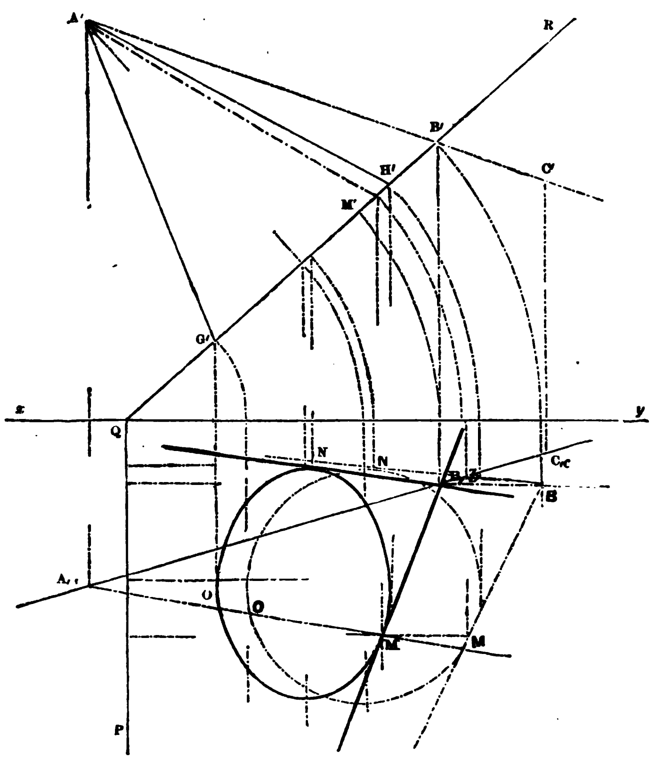
the given angle with the plane P Q R, and also being in a plane parallel to the second plane, they are parallel to that plane.

Problem XXVI.

To draw a plane to contain a given line, and to make a given angle with a given plane, not containing the line.

Let P Q (Fig. 70) be the trace of the plane, A C the plan of the line. Draw $x y$ at right angles to P Q; and on $x y$ as a line of level make an elevation Q R of the given plane.

Fig. 70.



Make any point A,a in the given line the vertex of a right cone with circular base standing on the plane P Q R; its gene-

ratrix making with that plane an angle equal to the given angle. Let A′ G′ H′ be the elevation of that cone, the ellipse M O N being the plan of its base. Find by Problem VII. B,*b* the point in which the line (A C, A′ C′) meets the plane P Q R. From B draw B M and B N, touching M O N in M and N respectively: B M and B N will be the plans of two lines in which the plane P Q R is cut by two planes, which will fulfil the conditions of the problem, when made to contain the point A *a*: which may readily be accomplished by determining the trace of the plane by means of those of the lines (A B, A′ B′) and (A M, A′ M′).

By turning the plane P Q R, with the base of the cone in it, about its trace P Q, as shown in the figure, and drawing tangents **B M**, **B N** to the circle **M O N**, the points M and N may be found (Chap. I. 36), without describing the ellipse M O N and drawing tangents to it.

Problem XXVII.

To draw a plane that shall have a given inclination, and make a given angle with a given plane.

The steps in the solution of this problem will be indicated without the aid of a diagram, which a knowledge of the preceding portion of the work will enable the reader to supply.

Make any point the common vertex of two right cones A and B, with circular bases; A having its axis perpendicular to the horizontal plane and its generatrix making with that plane an angle equal to the given inclination; B having its axis perpendicular to the given plane, its generatrix making with that plane an angle equal to the angle between it and the required plane.

Determine the intersections of these cones by the horizontal plane; that of A will be a circle; that of B an ellipse. Every common tangent that can be drawn to the circle and the ellipse will be the trace of a plain fulfilling the given conditions; and since the inclination of the plane is given, its scale can be constructed.

It is evident that, when the circle and the ellipse fall entirely without each other, four common tangents can be drawn to them;

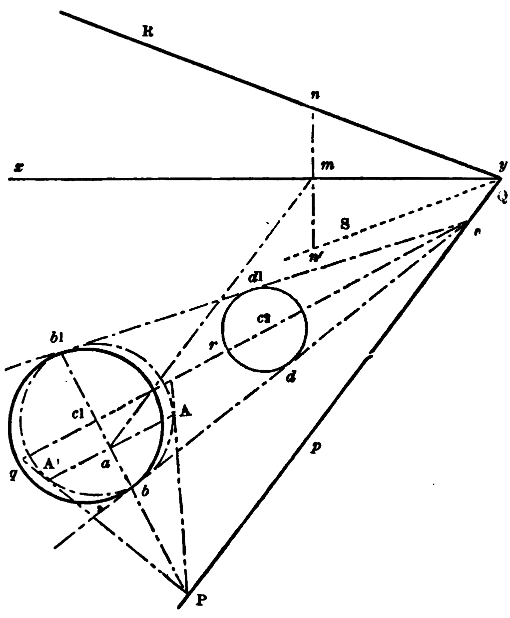
when they touch, three or one; when they cut, two; when one falls within the other without contact, none. The problem, therefore, admits of one, two, three, or four solutions; or it is impossible.

PROBLEM XXVIII.

To draw a tangent plane to two given spheres through a given point.

Assume the centre c_1, c_2 of the spheres (Fig. 71) and the given point p to be in the horizontal plane, $b\ q\ b_1$ and $d\ r\ d_1$ being great

Fig. 71.



circles. To these circles draw the common tangents $v\,d\,b$ and $v\,d_1 b_1$; the point v will lie on the line $q\ v$, passing through the centres c_1, c_2, and if the tangents revolve around this line $v\ q$ they will gene-

rate a cone, such that every plane which touches it will also touch the given spheres. The horizontal trace of one such plane will evidently be the straight line P Q passing through the given point p and the vertex of the cone v, which will be one point of contact. Join b_1 b and produce it to P; b_1 P will be the trace of a vertical plane containing a second point of contact, whilst b_1 b will be the diameter of the circle in which that plane cuts the sphere: turn this circle about its diameter b_1 b until it takes the position A b_1 A_1 in the horizontal plane: from P draw P A to touch this circle in A: A will be the point of contact: draw A a perpendicular to b_1 P; a will be the plan of that point. To determine the vertical trace of the plane: through a draw a m parallel to P Q, draw m n perpendicular to x y and equal to A a; n will be a point in the vertical trace of the plane, which will therefore be Q R, drawn through Q and n. It is evident that, if in n m produced m n' be made equal to m n, the straight line Q S drawn through n' will be the vertical trace of a second plane touching the cone, and therefore the spheres, in the point A$'$ below the horizontal plane.

When the circles b_1 g b and d_1 r d lie entirely without each other, as in the figure, two pairs of common tangents can be drawn, thus giving rise to four solutions. In the second pair the point of intersection will be between c_1 and c_2, but the construction will be similar to the preceding one.

When the circles touch there will generally be only two solutions; but a third, when the point p lies in the vertical plane passing through the point of contact.

When the spheres cut each other there will be only two solutions. When one sphere falls within the other there will be no solution.

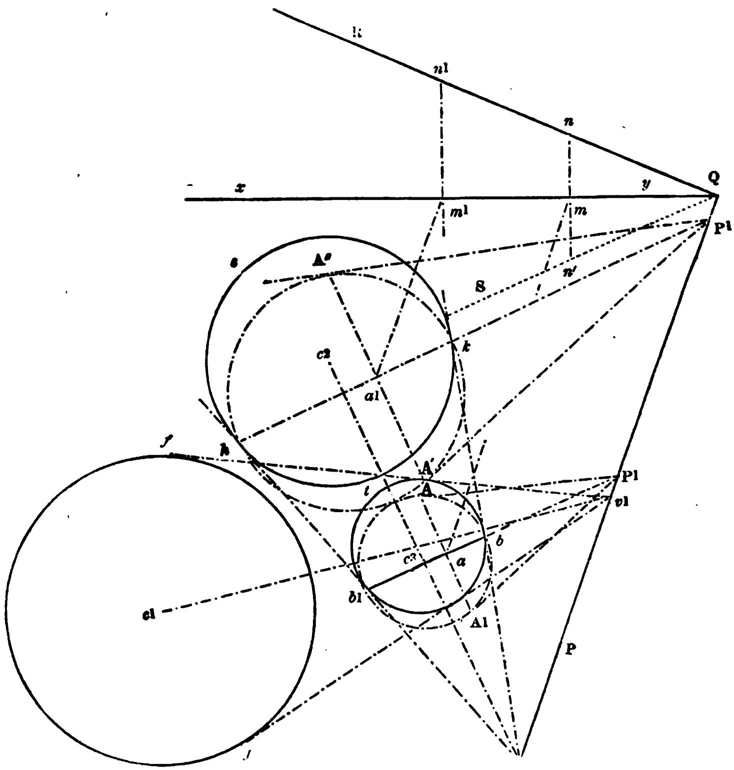
Problem XXIX.

To draw a tangent plane to three given spheres.

Assume the plane containing the centres c_1, c_2, c_3 (Fig. 72) of the spheres, as the horizontal plane of projection, so that f r g, h s k, b b_1 t are great circles. Let v_1 be the vertex of a cone circumscribing the spheres c_1, c_3: v_2 the vertex of a cone cir

cumscribing the spheres c_2, c_3, any plane which touches these cones will also touch the three spheres. Now the horizontal trace of a plane touching the cones is evidently the line P Q passing through v_1 and v_2. To determine the vertical trace it will be necessary to find the point in which the plane touches any

Fig. 72.



one of the spheres, and this may be accomplished in the same way as in Prob. XXVIII. The constructions shown in the figure for the spheres c_2, c_3, both give the same trace Q R. The vertical trace of a second plane, touching the spheres below the ground line, will be a straight line Q S such that the angle S Q x is

equal to the angle R Q x. This problem will manifestly admit of various solutions since the three spheres can be combined two and two in three ways, and the vertices of the cones may fall between the spheres as well as exterior to them.

Problem XXX.

To determine the plan of a rectilineal figure, having given (1) the inclination of its plane, and that of one side: (2) the inclination of two of its sides.

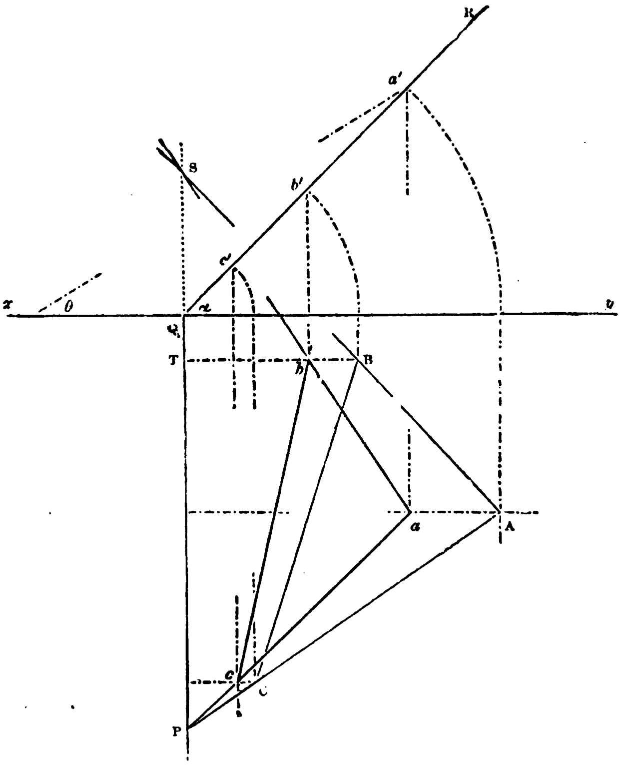
Either of these data will suffice to determine the position of the figure with reference to the horizontal plane. The problem will be simplified by assuming the plane of the figure at right angles to the vertical plane of projection; consequently, its trace will be perpendicular to the ground line, or line of level.

1. Let P Q R (Fig. 73) be the given plane, draw a P, the plan of a line in it, having the given inclination (Prob. XIII.). Find by Chap. I. 36, the position A, of the point whose plan is a, when the plane P Q R has been turned about its trace P Q into coincidence with the horizontal plane. Make A B equal to the side whose inclination is given; on A B describe the figure A B C in its real magnitude; find its plan a b c, by Chap. II., Prob. XXXIX., as shown in Fig. 73.

Note.—It is manifest that any line A B, and its plan a b, if produced, will meet the trace P Q in the same point S. The point b may, therefore, be found by drawing B T parallel to x y, and cutting a S in b; this method often considerably shortens the construction. Although introduced here as a separate problem, this problem is merely a combination of Prob. V. Chap. III. with Prob. XXXVIII. Chap. II.

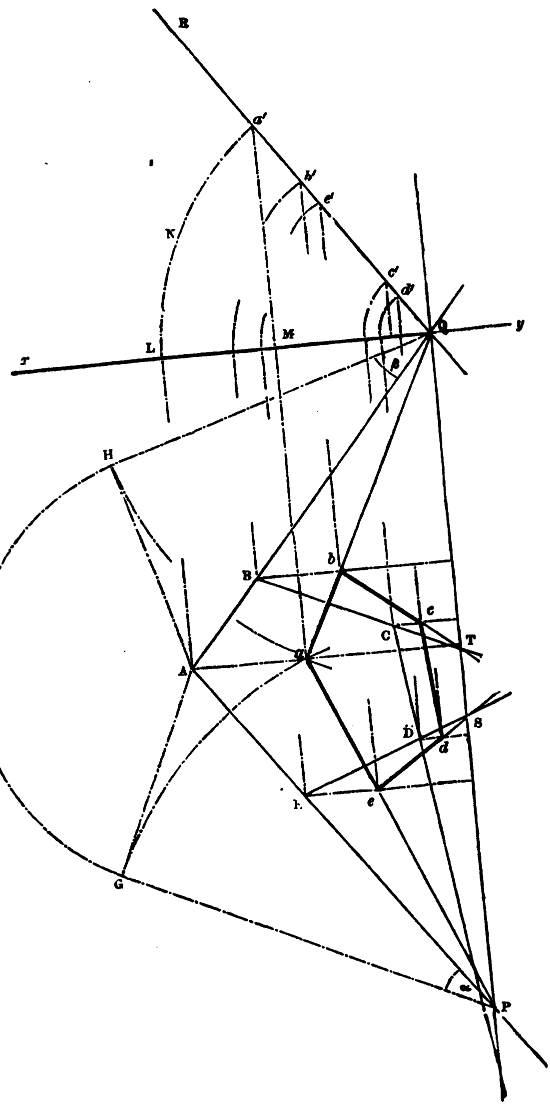
2. Let A B C D E (Fig. 74) be the given figure, its two sides A B and A E being inclined $\alpha°$ and $\beta°$ respectively to the horizon. In A E produced assume any point P, as the trace of A E; make the angle A P G equal to α; from A draw A G perpendicular to P G: with A as a centre and radius A G, describe the circle G K H; draw Q H touching this circle in H, making an angle β with A B, and meeting A B, or A B produced, in Q;

Fig. 73.



Q will be the trace of A B; and the straight line P Q the trace of the plane containing the given figure. Then, by Chap. I. 36, the plan of A will be in the straight line A T, drawn from A perpendicular to P Q; its distance from P Q being equal to the base of a right-angled triangle, whose hypothenuse is A T and perpendicular A H. This triangle may be constructed in the following manner:—With P, or Q, as a centre, and radius P G, or Q H, describe a circle cutting A T in *a*; *a* will be the plan of A. Through Q draw *x y* perpendicular to P Q; *x y* may be assumed as the trace of a vertical plane perpendicular to the

Fig. 74.



plane of the figure. Draw A L perpendicular to $x\,y$; with centre Q, and radius Q L, describe a circle L N a'; draw a M perpendicular to $x\,y$, and meeting this circle in a'; a' will be the elevation of A on the vertical plane, whose trace is $x\,y$. Through a' draw Q R; Q R will be the vertical trace of the plane of the figure; the angle R Q x its inclination to the horizon, and Q M a' the triangle required. The elevations b_1, c_1, d_1, e_1, and the plans b, c, d, e, may now be readily found (Chap. I. 36); or by the principle stated in the preceding Note. Both constructions are shown in the Figure.

Exercises.

1. Two lines perpendicular to each other, and each 3″ long, have a common extremity; draw their projections—

   a. When one line is parallel to $x\,y$, the other inclined at 30°.

   b. When inclined at 30° and 43° degrees respectively.

2. The horizontal trace of a plane makes an angle of 55°, and the vertical an angle of 35°, with $x\,y$; drawing these traces, draw the scale of slope of the plane and determine its inclination.

3. Draw a line, $a\,b$, 4″ long, its extremities being figured 6, 25, and a point p figured 11, 2″ distant from the middle point of $a\,b$; determine the plan of the line drawn from the point perpendicular to the line $A\,B$.

4. Draw the plans of three lines mutually at right angles, two of them being inclined at 27° and 35°.

5. Two planes contain a right angle: one is inclined at 60°, and their intersection at 50°; represent them by their scales of slope or by their traces.

6. Draw a triangle $A\,B\,C$, of 3, 3·5, 4 inches sides; the indices of A and B are 1·7 and 3·6; determine that of C, so that the triangle may be right-angled at A.

7. Determine the index of C (*Ex.* 6), the triangle being supposed right-angled at C instead of A.

8. Draw the scale of a plane inclined at 50°; in it place a line inclined at 25°; through this line draw another plane of which the horizontals are at right angles to those of the first plane. Determine the angle between the two planes, and the inclination of the second plane.

9. Draw a plane, inclined at 56°; in it place a straight line inclined at 46°; from any point in this line draw a perpendicular to the plane, 2 in. long.

10. Draw two planes inclined at 40° and 63° respectively and intersecting in a line inclined at 27°. Find the dihedral angle contained by the two planes.

11. Two lines, each two inches long, are inclined at 25° and 30° respectively to the horizon, in a plane inclined at an angle of 50°. Draw the plan of the lines in position, and construct the real angle contained by them. Scale, 10 units to an inch.

12. Determine the inclination to the horizon of a line 2·78 inches long, one extremity being 1·56 inches higher than the other. Also of another line 3·6 inches long, its plan being 2·3 inches long.

13. A line, 2 inches long, has its ends respectively 5 and 19 units above the horizontal plane. Find its intersection with a plane inclined at 60° to the horizon, assuming any relative position on plan. Scale, 10 units to one inch.

14. Find the intersection of two planes inclined at 30° and 54° to the horizontal; their horizontals being parallel.

15. The plans of two lines contain an angle of 80°, the lines being inclined to the horizon at angles of 50° and 35°. Determine the real angle contained by the lines, and also the inclination of the plane in which they lie. Through the line inclined at 35°, draw a plane making an angle of 75° with the plane containing the two lines.

16. Draw a plane, making an angle of 42° with another inclined at 50° to the horizon; and passing through a line, not in the given plane, inclined at 30° to the horizon.

17. Draw a line inclined 35° to the horizon; through it draw

a plane inclined at 58°; and in the latter place a line making an angle of 30° with the former line.

18. Draw a plane inclined 42° to the horizon, and a perpendicular to it 2 inches long: through the perpendicular draw a plane, making an angle of 55° with the horizon. Find by construction the dihedral angle of the planes.

19. A plane inclined at 60° to the horizon makes with another plane an angle of 70°, the intersection of the two planes being inclined at 42°. Find the inclination of the second plane.

20. An equilateral triangle, with a side of 2·2 inches, has two of its sides inclined at 25° and 40° to the horizon. Draw its plan. Determine the inclination of the plane of the triangle. Draw the plan of the circle circumscribing the triangle.

21. Through a line inclined at 40° to the horizon, draw a plane, making an angle of 30° with a plane inclined at 25° to the horizon. Scale, 10 units to an inch.

22. An equilateral triangle of 3 inches side rests on one angle and has the sides containing that angle inclined at 20° and 30° respectively to the horizon. Construct its plan.

23. Construct the plan of a square of 2 inches side resting on a plane inclined at 50° to the horizon, one of the sides being inclined at an angle of 35° to the horizon.

24. Draw the plan of an isosceles triangle having a base of 1·5 inches and sides of 2·25 inches; the triangle being so placed that the base is inclined 23°, and the line joining one end of the base with the middle point of the opposite side 51°, to the horizon.

25. A regular pentagon A B C D E of 1·5 inches side has the corners A, C, E, 1 inch, 1·5 inches, and 1·8 inches, above the H. P.; draw its plan, and an elevation on a plane parallel to the diagonal C E.

26. Construct a triangle abc; $ab = 10$, $ac = 15$, $bc = 20$ units: bisect bc in d, and ac in e; join de. The points a, b, c are respectively 13, 23, 33 units high: figure two points in de, so that it may be 10 units above the plane of abc (unit $= \cdot 1$ inch).

27. Draw the traces and scale of slope of a plane inclined at 60°.

28. An oblique cone 30 units high, radius of base 10 units, its vertex vertically over a point in the circumference of its base, is to be represented lying on the H. P. on its longest slant side.

29. One of the lines containing an angle of 30° is horizontal, the plan of the angle measures 50°: determine the inclination, and draw the scale of slope, of the plane in which it lies.

30. Draw an isosceles triangle, the equal sides being each 2·5 inches, the base 3·5 inches; figure the base 12, 30; the vertex 19; determine the true form of the triangle and the inclination of its plane.

CHAPTER IV.

THE PROJECTION OF SOLIDS.

1. The determination of the projection of a solid, when bounded by planes, will evidently consist in finding the projections of its edges, by means of those of its angular points. Should, however, any of its bounding superficies be curvilinear figures, their projections must be constructed in a manner similar to that shown in Chap. II. Prob. XL.

2. *A polyhedron* is a solid figure bounded by plane rectilineal figures.

A regular polyhedron has all its faces equal, similar, rectilineal figures, and all its solid angles equal. Of such solids there are *five only*. These are:—

(α.) *The tetrahedron*, bounded by four equilateral triangles.
(β.) *The hexahedron* or *cube*, bounded by six squares.
(γ.) *The octohedron*, bounded by eight equilateral triangles.
(δ.) *The dodecahedron*, bounded by twelve regular pentagons.
(ϵ.) *The icosahedron*, bounded by twenty equilateral triangles.

For definitions of other solids mentioned in this chapter, see *Euc.* XI., *definitions.*

Problem I.

To construct the projections of the tetrahedron.

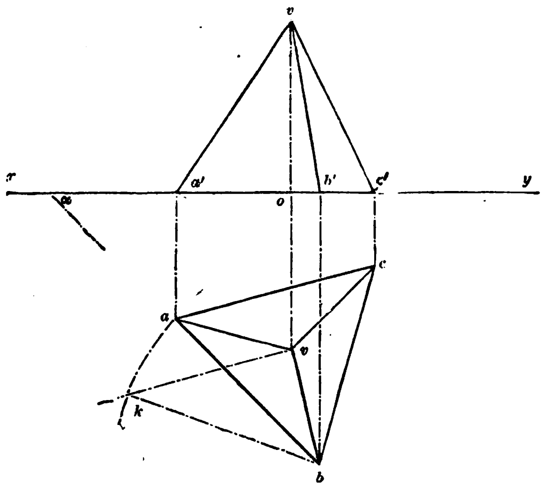
Assume the solid to be resting on one of its faces on the horizontal plane,

Let $a\ b$ (Fig. 75) be one of the edges resting on the horizontal plane, and inclined at any given angle a to $x\ y$; on $a\ b$ describe the equilateral triangle $a\ b\ c$; find v the centre of the circumscribing circle; join $a\ v$, $b\ v$, $c\ v$; v will be the plan of the vertex;

a b c will be that of the base; *a v*, *b v*, and *c v*, those of the edges meeting in *v*.

From *v* draw *v k* perpendicular to *v b*; with *b* as a centre, and

Fig. 75.



radius *b a*, describe a circle cutting *v k* in *k*; *k v* will evidently be the perpendicular height of the pyramid. Draw *a a'*, *b b'*, *c c'*, *v v'* perpendicular to *x y*; make *o v'* equal to *v k*; join *a' v'*, *b' v'*, *c' v'*; the figure thus formed will be the elevation of the solid.

Problem II.

To construct the projections of the cube.

(*a*) Having one face horizontal: (*b*) having a face and one of its edges inclined at given angles: (*c*) having two contiguous edges inclined at given angles: (*d*) having two faces inclined at given angles.

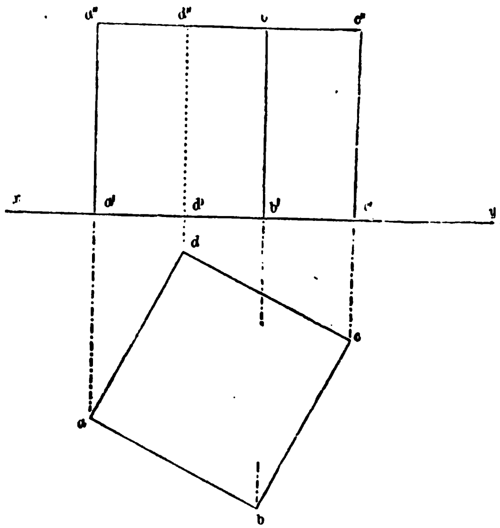
(*a*) Assume the solid to be resting on one of its faces on the horizontal plane.

Let *a b* (Fig. 76) be one edge of that face. Describe on *a b*,

in the horizontal plane, the square $a\ b\ c\ d$; this will be the required plan.

Draw $a\ a'$, $b\ b'$, $c\ c'$, $d\ d'$, perpendicular to $x\ y$; in these lines produced make $a'\ a''$, $b'\ b''$, $c'\ c''$, $d'\ d''$, each equal to $a\ b$. The figure $a''\ c''\ c'\ a'$ will be the elevation of the solid. It would,

Fig. 76.



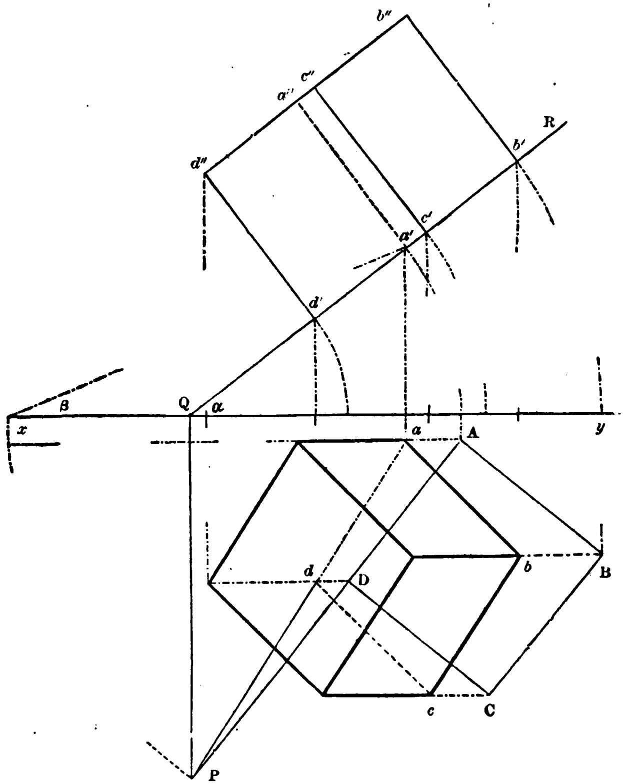
perhaps, be simpler to make $a'\ a''$ equal to $x\ y$, and through a'' to draw $a''\ c''$ parallel to $x\ y$.

(*b*) Let the face A B C D (Fig. 77) be inclined at an angle α to the horizon, and the edge A B at an angle β; α 7 β.

On $x\ y$, a line of level perpendicular to the trace P Q of the plane of the base, make an elevation Q R of this plane; in it place the line A P, whose plan is a P, inclined at an angle β (Chap. III. Prob. V.); construct the elevation $d'\ a'\ c'\ b'$ of the base, by turning the square A B C D through the angle α, thence find its plan $a\ b\ c\ d$. The elevations of the edges perpendicular to the plane P Q R will be perpendicular to Q R (Chap. I. 31), and equal in

length to the edge of the cube (Chap. I. 13) : draw $d'\ d''$, $a'\ a''$, $c'\ c''$, $b'\ b''$ perpendicular to Q R and equal to A B : join $b''\ d''$: the figure $b'\ d''$ will be the elevation, the plans of the points a'', b'', c'', d'' will be the points in which perpendiculars to $x\ y$ drawn

Fig. 77.



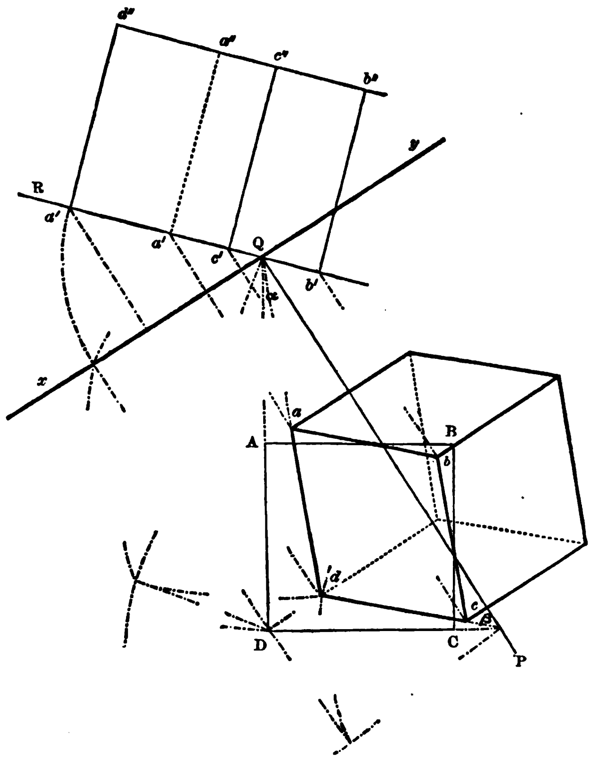
from these points cut the parallels to $x\ y$ drawn from A, B, C, D. The construction is shown in the figure.

(*c*) Let the edges D A, D C (Fig. 78) be inclined at angles α and β to the horizon.

On $x\ y$, as a line of level, perpendicular to P Q, the horizontal

trace of the plane containing the edges D A, D B determined by Prob. XXX. Chap. III. make an elevation b', c', a', d' of the base A B C D; thence determine the plan $a\ b\ c\ d$, and complete the plan and elevation of the solid as in the preceding case.

Fig. 78.

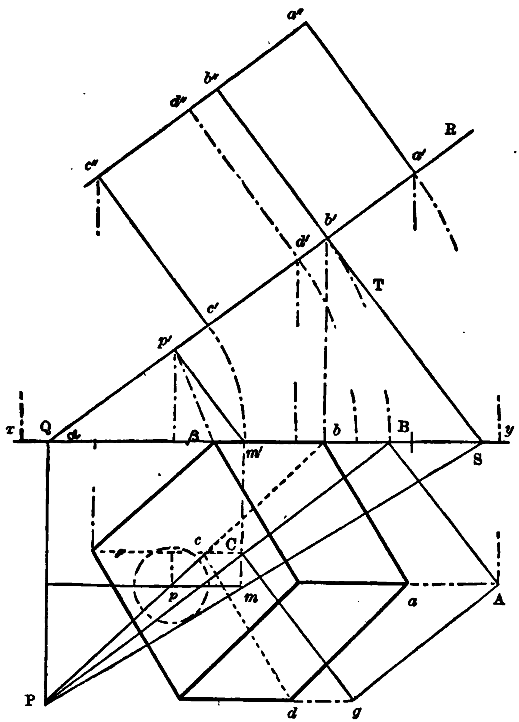


(*d*) Let two contiguous faces be inclined at angles α and β (Fig. 79).

Assume the plane of the face, inclined at α to the horizon, to be perpendicular to the vertical plane of projection, so that its trace P Q is perpendicular to $x\ y$: take any point (p, p') in this plane as the vertex of a right cone, whose generatrix makes an angle

β with the horizontal plane on which it stands: draw the straight line ($p\ m$, $p'\ m'$) perpendicular to the plane P Q R, find its trace m: from m draw the tangent P m S to the base of the cone: this will be the trace of a plane inclined at β, and perpendicular to P Q R (see Prob. XXIII. Chap. II.) The line P b is evidently the

Fig. 79.



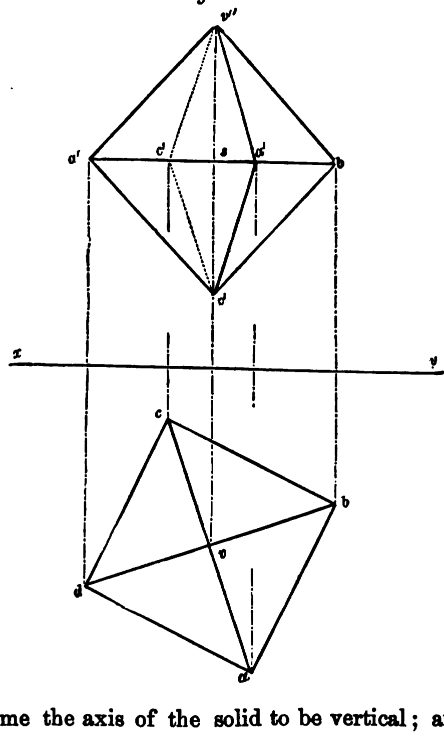
plan of the intersection of the planes P Q R, P S T, and therefore contains the plan of the edge between the faces whose inclinations are given: let P B be this intersection turned down into the horizontal plane; make B C equal to the edge of the cube: on it describe the square B C D A, and complete the projections as in the two preceding cases.

Problem III.

To construct the projections of the octohedron.

(*a*) When its axis is vertical: (*b*) when it lies on its face on the horizontal plane: (*c*) when two of its axes are inclined at given angles.

Fig. 80.



(*a*) Assume the axis of the solid to be vertical; and let *a b*, making an angle *α* with *x y* (Fig. 80) be one edge; on it describe the square *a b c d* : draw the diagonals *a c*, *b d* intersecting in *v*; this figure will be the plan of the solid, *v* being that of the vertex.

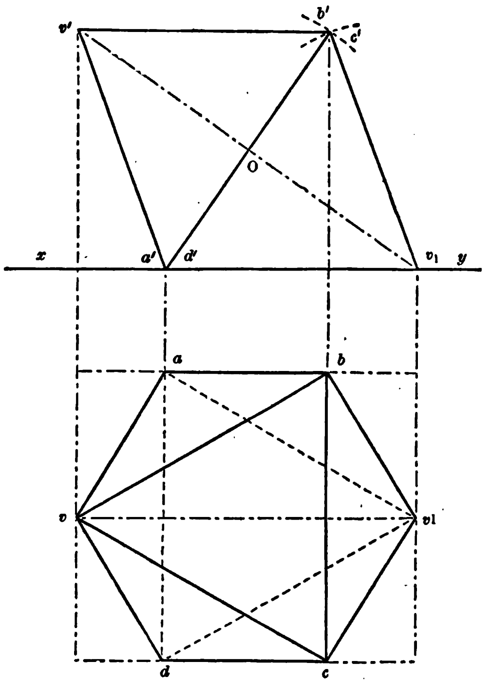
From *v* draw *v v''* perpendicular to *x y*; make *v' v''* equal to the diagonal of the square *a b c d* bisect *v' v''* in *s*; draw *d' b'*

parallel to $x\,y$, and $a\,a'$, $b\,b'$, $c\,c'$, $d\,d'$ perpendicular to $x\,y$; $v'\,v''$ a', b', c', d', will be the elevations of the angular points of the solid, whose elevation will be formed by joining $v''\,d'$, $v''\,a'$, $v''\,b'$, $v'\,d'$, $v'\,a'$, $v'\,b'$: the elevations $v''\,c'$ and $v'\,c'$ would be invisible.

(*b*) Let the solid lie on one of its faces on the horizontal plane.

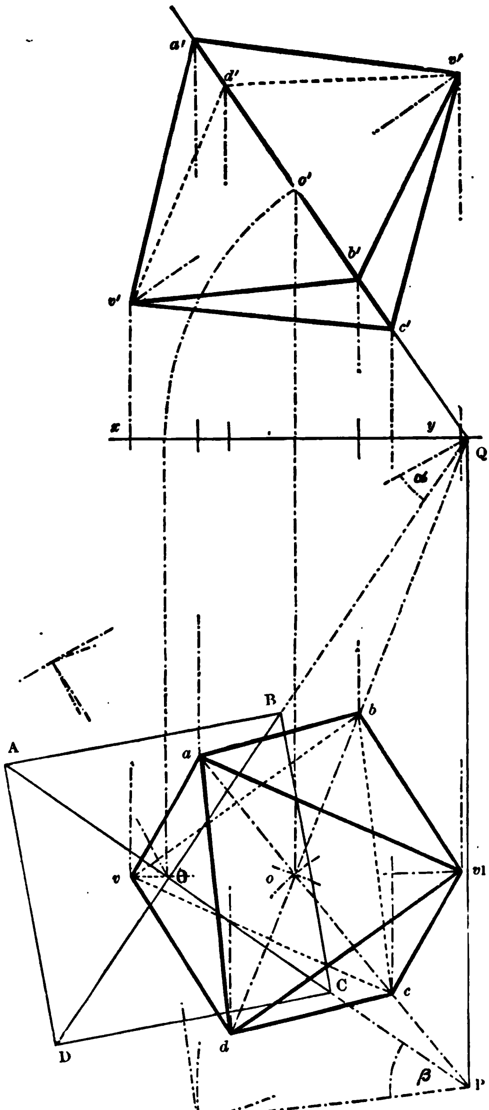
Let $a\,d\,v_1$ be the face in the horizontal plane, and $x\,y$ parallel to the axis passing through v; then if $a\,d'$ and $v_1\,v'_1$ are drawn

Fig. 81.



perpendicular to $x\,y$, $a'\,v'_1$ is the elevation of the face $a\,d\,v_1$ Make $a'\,b'$ equal to an edge of the solid, $b'\,v'_1$ equal to $d'\,v'_1$; draw $v'_1\,v'$ perpendicular to $a'\,b'$ and bisected in O, join $a'\,v'$ and $b'\,v'$, thus completing the elevation. The plan may be at once constructed as shown in the figure.

Fig. 82.



(*c*) Let two of the axes be inclined at $\alpha°$ and $\beta°$.

Find P and Q, the traces of the diagonals D B, A C, of the square A B C D, which are inclined $\alpha°$ and $\beta°$ respectively, make an elevation P Q R of the plane of the base A B C D (Chap. III. Prob. XXX.) on $x\ y$ at right angles to P Q: from o, the plan of the intersection of the diagonals D B, A C, draw $o\ o'$ perpendicular to $x\ y$, and $v'_1\ o'\ v'$, perpendicular to Q R; make $o'\ v'$ equal to $o\ v'_1$, equal to the semidiagonal of the square, v' and v'_1 will be the elevations of the vertices of an axis. Through o draw $v\ o\ v_1$ perpendicular to P Q: draw $v'\ v$ and $v'_1\ v_1$ perpendicular to $x\ y$, v and v_1 being thus determined: join those points with a, b, c, d, the angular points of the plan of the square A B C D; this completes the plan. The elevation may then be constructed as usual. The solution depends upon the principle (Chap. I. 31) that the axis, perpendicular to the base A B C D, being perpendicular to the plane P Q R, its projections will be respectively perpendicular to P Q and Q R.

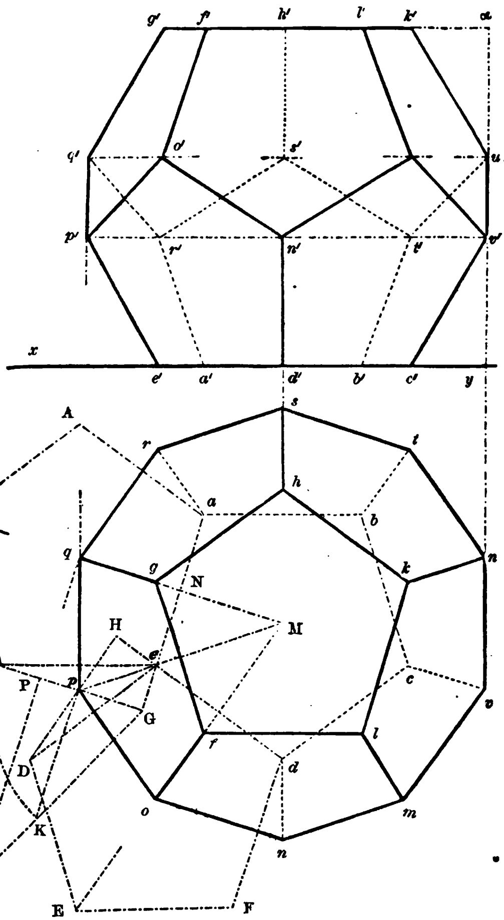
Problem IV.

To construct the projections of the dodecahedron.

Assume the solid to rest on its base, $a\ b\ c\ d\ e$ (Fig. 83), on the horizontal plane, with the edge $a\ b$ parallel to $x\ y$: then if g, h, k, l, f be the middle points respectively of the arcs of the circle circumscribing the pentagon $a\ b\ c\ d\ e$, the pentagon $g\ h\ k\ l\ f$ will be the plan of the top face. Turn the pentagons, whose bases are $a\ e$ and $e\ d$, about $a\ e$ and $e\ d$ until they take the positions a A B C e and e D E F d in the horizontal plane. Draw through C and D vertical planes, whose traces C G and D H are perpendicular to $a\ e$ and $d\ e$ respectively; the point p in which these traces meet will evidently be the plan of the common extremity of the edges e C and e D, and will be situated on the radius M e produced, M being the centre of the circle before mentioned. With centre G and radius G C describe a circle, from p draw p K perpendicular to G C, and meeting this circle in K, p K will be the height of the point, whose plan is p, above the horizontal plane; join G K and produce it to L, so that G L is equal to N B, the height of one of the pentagons; draw L P

perpendicular to G C: L P will be the height of the point B above the horizontal plane, that is the height of the top

Fig. 83.



face, and G P will be the distance of the plan of the point B from N measured along M g: if, therefore, N q be made equal to G P, q will be the plan of the point B, and the line

$p\ q$ will be the plan of C B : the plans of the remaining angular points of the solid may be found in the same way; but since the plans of all the edges similarly situated with C B, B A form a regular decagon, if a circle be described with M as a centre and radius M g, and straight lines $q\ r$, $r\ s$, $s\ t$, $t\ u$, $u\ v$, $v\ m$, $m\ n$, $n\ o$, $o\ p$, each equal to $p\ q$, be placed round in it, and the angular points of this figure be joined with the corresponding points of the top and bottom faces, the plan will be completed. To draw the elevation: on $u\ a$, drawn perpendicular to $x\ y$, make $y\ v'$ equal to p K, $y\ u'$ equal to P L, and $u'\ a$ equal to $y\ v'$, through a, u', v^1 draw straight lines parallel to $x\ y$, the elevations of all the angles of the solid which are all situated on one or other of the lines $a\ g'\ u'\ q'\ v'\ p'$, $x\ y$ may be found in the usual way.

Problem V.

To construct the projections of the icosahedron.

Assume one axis of the solid to be vertical, and one of its edges parallel to the vertical plane of projection.

Let $a\ b$ (Pl. I. Fig. 6) be one edge of the solid; on $a\ b$ describe the regular pentagon ($a\ b\ c\ d\ e$, $e'\ a'\ d'\ b'\ c'$) in a horizontal plane, so that $e'\ c'$ is parallel to $x\ y$. Determine a point (s, s') such that by joining it with the angular points of the pentagon $a\ b\ c\ d\ e$, five equilateral triangles will be formed. To effect this, s being the centre of the pentagon, describe on $s\ b$ the right-angled triangle $s\ b\ n$, in which the hypothenuse $b\ n$ is equal to $b\ c$; $s\ n$ will be the perpendicular height of a pyramid whose base is the pentagon ($a\ b\ c\ d\ e$, $e'\ a'\ d'\ b'\ c'$). In a second horizontal plane describe the pentagon $g\ h\ k\ l\ m$ equal to and concentric with the former, but having its angular points respectively in the middle points of the arcs of the circumscribing circle subtended by the sides of the pentagon $a\ b\ c\ d\ e$. To determine the distance of this second horizontal plane from the former one, join $m\ e$, describe on it the right-angled triangle $m\ e\ o$, having its hypothenuse $m\ o$ equal to $m\ l$: $e\ o$ will be equal to the vertical distance between the two planes.

Join $s\ c$, $s\ k$, $s\ d$, $s\ l$, $s\ e$, $s\ m$, $s\ a$, $s\ g$, $s\ h$, $a\ g$, $g\ b$, $b\ h$, $h\ c$, $c\ k$, $k\ d$, $d\ l$, $l\ e$, $m\ a$; this will complete the plan of the solid.

To finish the elevation, make d' s' equal to s n; join s' e', s' a', s' d', s' b', s' c'; this will be the elevation of the pyramid whose base is a b c d e: in the perpendicular e e' make e' m' equal to e o, draw through m' a straight line parallel to x y and meeting the perpendiculars from l, d, k, and h in l', g', k', h': m' l' g' k' h' will be the elevation of the pentagon g h k l m: join a' m', a' g', b' g', b' h', l' e', l' d'; k' d', k' c'; this will complete the elevation of a zone bounded by ten equal equilateral triangles (constr.). The elevation of the solid may now be completed by drawing that of the pyramid whose base is g h k l m, and vertex s', s'_1.

PROBLEM VI.

To construct the projections of a pyramid lying on one of its faces on the horizontal plane.

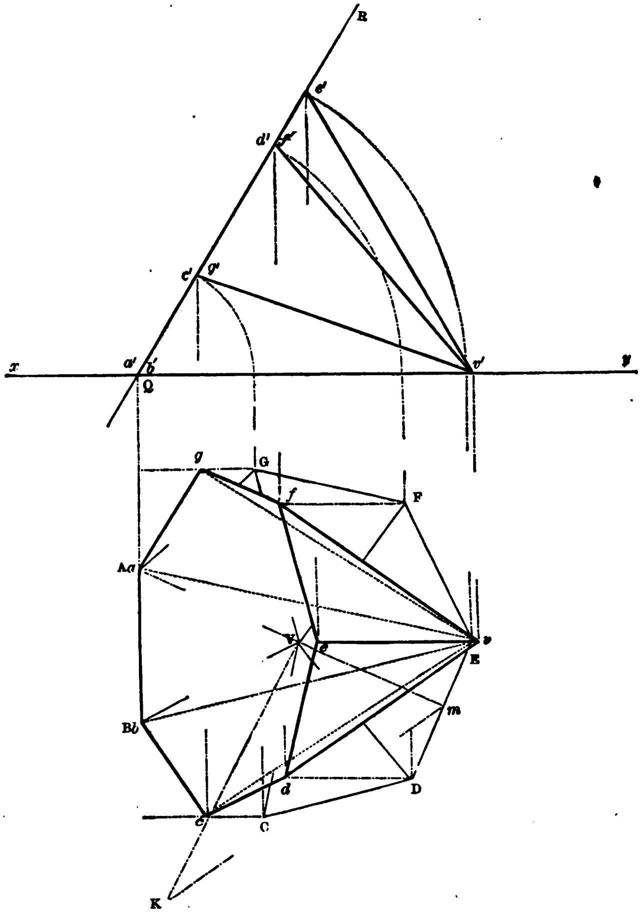
Suppose the pyramid to be a heptagonal one, having its axis parallel to the vertical plane of projection, and consequently the plane of its base perpendicular to that plane.

Let A B C D E F G, V (Fig. 84) be the plan of the figure standing on its base on the horizontal plane; A B being the side of the base in contact with the horizontal plane, and, when produced, meeting x y at right angles in the point Q: B Q will be the horizontal trace of the plane of the base.

Draw V m perpendicular to D E: V K perpendicular to V m, and equal to the perpendicular height of the pyramid: join m K: m K will be the slant height, and V m K the profile angle between any face and the base; and therefore the measure of the inclination of the base to the horizontal plane.

Through Q draw a straight line Q R, making with x y an angle R Q y, equal to V m K: Q R will be the vertical trace of the plane containing the base. The elevation of the base, a' c' d' e', may now be found by Prob. XXIX. Chap. II. Make a' v' equal to m K; join v' e', v' d', v' c'; this will complete the elevation of the solid. The plans a, b, c, d, e, f, g and v, corresponding to the elevations a', b', c', d', e', f', g', and v', will be found by drawing through the elevations perpendiculars to x y (Chap. I. 30), and through A, B, C, D, E, F, G and V parallels to x y.

Fig. 84.



PROBLEM VII.

To construct the projections of a prism; having given the inclination of the plane of its base, and that of one edge.

Let the prism be a hexagonal one: its base being the regular

hexagon, A B C D E F (Fig. 4, Pl I.), assume the plane P Q R, containing the base, to be perpendicular to the vertical plane of projection: the plan, $a\ b\ c\ d\ e\ f$, of the base A B C D E F, and its elevation $a'\ b'\ f'\ c'\ e'\ d'$, may be found at once by Prob. XXX. Chap. III. The construction is omitted in the figure to avoid rendering it more complex.

From a', b', c', d', e', f', draw $a'\ a'_{,,}$ $b'\ b'_{,,}$ &c. &c., perpendicular to Q R: make $a'\ a'$, equal to the height of the prism; through a'_1, draw $a'_{,}\ b'_{,}\ f'_{,}\ c'_{,}\ e'_{,}\ d'_{,,}$ parallel to Q R; this will complete the elevation of the prism.

To complete the plan it will be necessary to observe that the edges of the solid, being perpendicular to the plane P Q R, will be projected in straight lines perpendicular to P Q, the trace of that plane (Chap. I. 31). If therefore a straight line be drawn through a perpendicular to P Q, and one through $a'_{,,}$ perpendicular to $x\ y$, the point $a_{,,}$ in which these lines cut each other, will be the plan of the point whose elevation is $a'_{,}$. The plans of the other angular points of the upper end of the prism may be determined in a similar manner, and the plan of the solid will be completed by joining these points. Since, however, the projections of parallel lines are parallel, after the point a' has been found, b' may be found by drawing through a a straight line parallel to $a\ b$ and meeting B b produced in $b_{,}$; a similar construction will give the other points.

To construct an elevation of the prism on a third plane of projection, at right angles to the other two.

Let $v\ u\ v'$ be this plane: turn it about the trace $u\ v'$ into coincidence with the vertical plane of projection, as shown in the figure.

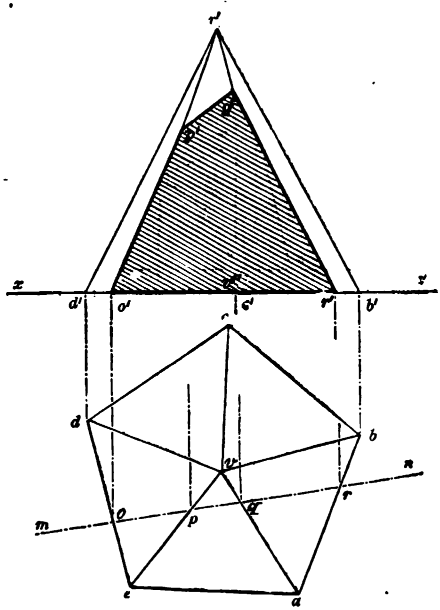
Let A m, perpendicular to $u\ v$, meet $u\ v$ in m: with u as a centre and radius $u\ m$, describe a circle cutting $x\ y$ in m': through m' draw a straight line perpendicular to $x\ y$, and through a' and $a'_{,,,}$ straight lines parallel to $u\ y$; the points $a_{,,}$ and $a''_{,,}$ in which these two lines respectively cut the former one, will evidently be the elevations of a and $a_{,,}$ on this third plane. The other points may be found in the same way.

PROBLEM VIII.

To construct the sectional elevation and plan of a pyramid standing on its base on the horizontal plane.

Let the pyramid be a pentagonal one, $b\ c$, one edge of the base being inclined at an angle a to $x\ y$ (Fig. 85); on $b\ c$ describe the

Fig. 85.

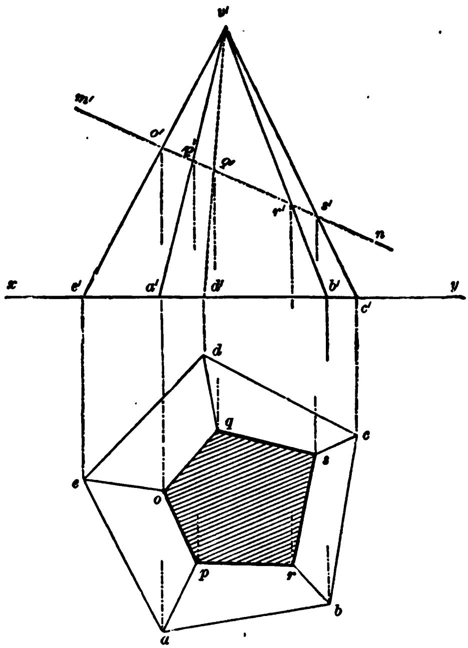


regular pentagon $b\ c\ d\ e\ a$: find v the centre of the circumscribed circle: join $v\ a$, $v\ b$, $v\ c$, $v\ d$, $v\ e$; this will be the plan of the pyramid. Draw $v\ v'$ perpendicular to $x\ y$ and make $v''\ v'$ equal to the perpendicular height of the pyramid: draw $a\ a'$, $b\ b'$, $c\ c'$, $d\ d'$, $e\ e'$ perpendicular to $x\ y$: join $a'\ v'$, $b'v'$, $c'\ v'$, $d'\ v'$, $e'\ v'$; this will complete the elevation of the pyramid.

(1.) To construct the elevation of a section made by a vertical plane whose trace $m\ n$ makes an angle β with $x\ y$, let this trace

cut *d e*, *e v*, *a v*, *a b* in the points *o*, *p*, *q*, *r* respectively. Then *o* and *r* being points in the horizontal plane, their elevations will be in the ground line; draw *o o′* and *r r′* perpendicular to *x y*; *o′* and *r′* (Chap. I. 30) will be the elevations of *o* and *r*: also *p* and *q* are the plans of points in the straight lines whose elevations are *a′ v′* and *b′ v′*; if therefore from *p* and *q* straight lines be drawn

Fig. 86.



perpendicular to *x y* and meeting *a′ v′* and *b′ v′* in *p′* and *q′*: *p′* and *q′* will be the elevations corresponding to *p* and *q*. Join *o′ p′*, *p′ q′*, *q′ r′*: the figure *o′ p′ q′ r′* will be the elevation required.

(2.) Let the section be made by a plane perpendicular to the vertical plane: its trace *m′ n′* making an angle δ with *x y* (Fig. 86), it is required to construct the plan of this section. Let the trace *m′ n* meet the elevations *e′ v′*, *a′ v′*, *d′ v′*, *b′ v′*, *c′ v′*

in the points o', p', q', r', s' respectively. The plans o, p, q, r, s, will be found by drawing $o'\,o, p'\,p, q'\,q, r'\,r$ and $s'\,s$ perpendicular to $x\,y$ (Chap. I. 30), and meeting $e\,v, a\,v, d\,v, b\,v, c\,v$ in o, p, q, r, s respectively. The figure $o\,p\,r\,s\,q$ will be the plan of the section.

NOTE.—When in (1) the cutting plane is parallel to the vertical plane of projection, its trace, the line $m\,n$, will be parallel to $x\,y$: the elevation will then evidently show the section in its real magnitude. This would be the same as what is called a sectional elevation on the line $m\,n$.

Obs.—The plan of the entire solid is shown in Fig. 85; its elevation in Fig. 86.

PROBLEM IX.

To construct the projections of a rectangular parallelopiped, having given the inclination of the plane of one face, and the plan on one side of that face.

Let P Q (Pl. I. Fig. 5) be the horizontal trace of the given plane, a its angle of inclination: $a\,b$ the given plan of one edge, m and n the lengths of the edges coterminous with that edge.

Find (Chap. II. Prob. XXIV.) Q R′ the vertical trace of the given plane: let the profile plane H R R′ employed in this construction pass through the point b, and turn it about its trace H R until it coincides with the horizontal plane (Chap. I. 36). Let H R R″ be the profile plane in this position: M R″ being its intersection with the given plane.

Draw b B perpendicular to H R: with centre M and radius M B describe a circle cutting H R in B′: B′ will be the position of the point, whose plan is b, when the plane P Q R′ has been turned about P Q into coincidence with the horizontal plane; join a B′; a B′ will be the real magnitude of the edge given by its plan $a\,b$ (Chap. I. 36). Draw a D perpendicular to a B′ and equal to m: the plan of D, which is the point d, will be found at once by the construction shown in the Figure (Chap. I. 36). Complete the parallelogram $a\,b\,c\,d$; this will be the plan of the face in contact with the plane P Q R′. The edge perpendicular to this face and terminated in the point (b, b') being perpendicular to the plane P Q R′ will be projected in R H, which is at right angles to P Q (Chap. I. 31); whilst on the profile plane this edge

will be represented in its real length B F, equal to n and perpendicular to M R$''$: draw F f perpendicular to H R; $b\,f$ will be the plan of the edge in question. Then, since the projections of parallel straight lines are parallel, the plan of the solid may be completed by drawing the parallelograms $a\,b\,f\,e$, $e\,a\,d\,h$, $g\,h\,d\,c$, $g\,f\,e\,h$, $b\,f\,g\,c$.

To construct the elevation. Since the lines whose plans are $a\,d$ and $c\,d$ lie in the plane P Q R$'$, their elevations will be determined by drawing vertical planes through $a\,d$ and $c\,d$: the point d' in which these elevations intersect will be the elevation of d: draw $c\,c'$ perpendicular to $x\,y$ cutting $d'\,c'$ in c'; c' will be the elevation of c: complete the parallelogram $a'\,b'\,c'\,d'$. The four edges perpendicular to the plane P Q R will have their elevations at right angles to Q R$'$ (Chap. I. 31): through $a'\,b'\,c'\,d'$ draw straight lines perpendicular to Q R$'$; and through e, f, g, h draw perpendiculars to $x\,y$ meeting the perpendiculars to Q R$'$ in e', f', g', h'; these points will be the elevations of e, f, g, h (Chap. I, 30). Join $e'\,f'$, $f'\,g'$, $g'\,h'$, $h'\,e'$; these lines will complete the elevation of the solid, and should be parallel respectively to $a'\,b'$, $b'\,c'$, $c'\,d'$, $d'\,a'$.

Problem X.

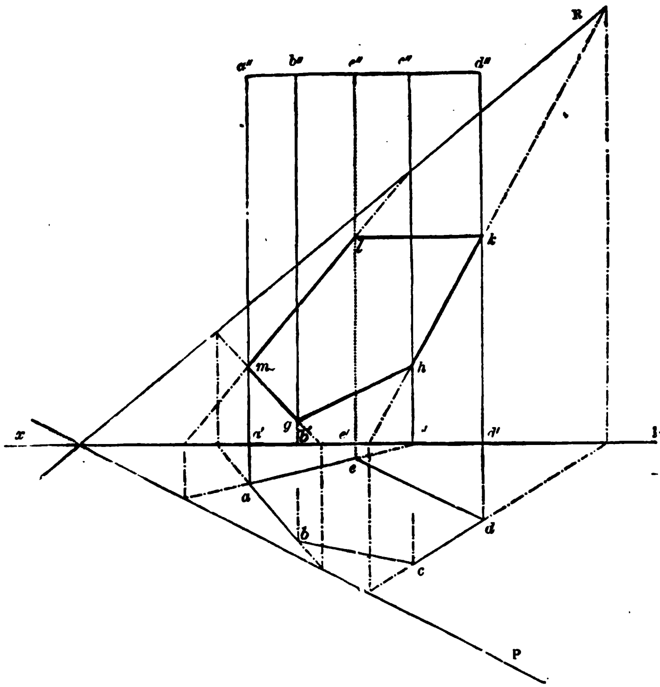
To determine the projections of the intersection of a right prism by a given plane, the real magnitude of the section, and its development.

The section of a prism by a plane is a polygon whose sides are the lines in which the faces of the prism are cut by the plane and whose angular points are the points in which the edges of the prism meet the plane.

When the prism is vertical the plan of the section will be the base, and the problem reduces itself to Prob. XXXVIII. Chap. II. viz. given the plan of a polygon, to find its elevation and real magnitude. The construction for determining the elevation by Prob. IX. Chap. II. is shown in Fig 87, where $a\,b\,c\,d\,e$ is the base of the prism and the plan of the section: $g\,h\,k\,l\,m$ the elevation of the section. The real magnitude may be readily found by the above-named problem.

The development of the prism will evidently be a rectangle whose base is equal to the sum of the sides of the figure $a\ b\ c\ d\ e$, and whose height is the height of the prism. Let A A′ be this rectangle, so that $AB = ab$; $BC = bc$; $CD = cd$; $DE = de$;

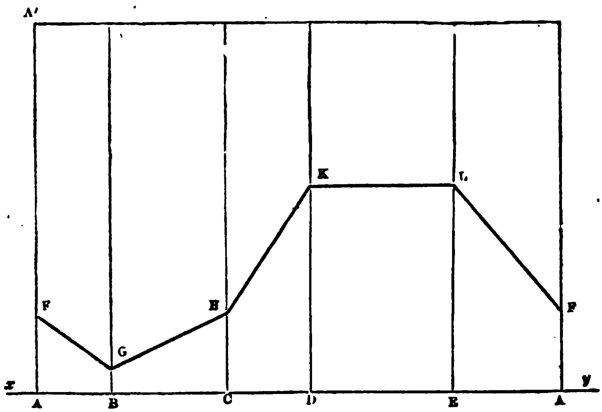
Fig. 87.



$EA = ea$. Through B, C, D, E draw perpendiculars to $x\,y$; make $A\ F = a'\,m$: $B\,G = b'\,g$; $C\,H = c'\,h$; $E\,L = e'\,l$: the line F G H K L F will be the development of the section (Fig. 88).

NOTE.—The intersection of a right cylinder by a given plane may be determined by a similar construction. This and some of the following problems may be simplified by assuming the trace P Q perpendicular to $x\,y$.

Fig. 88.



PROBLEM XI.

To find the projections of the intersection of a pyramid by a given plane: to determine the real magnitude of the section and its development.

The section of a pyramid by a plane is a polygon whose sides are the intersections of the faces by the plane, and whose angular points are the points in which the edges of the pyramid meet the plane.

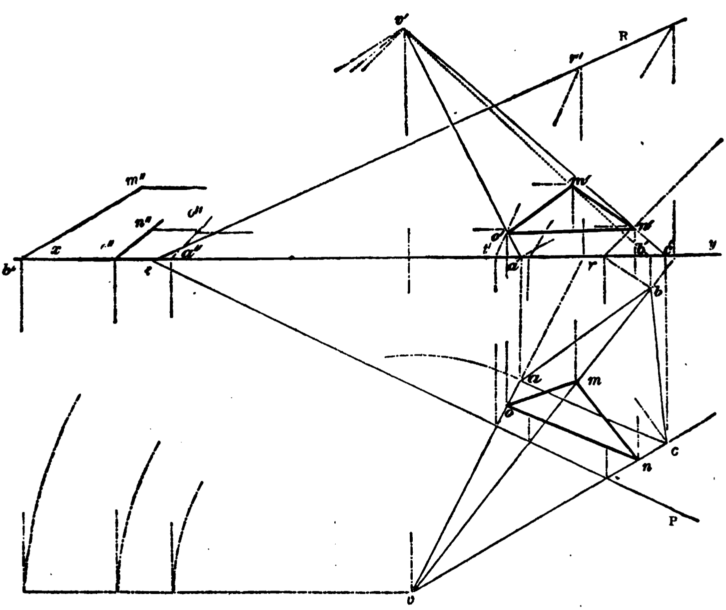
Let P Q R (Fig. 89) be the given plane; *a b c* the base of the pyramid, which is supposed to stand on its base on the horizontal plane, (v, v') being the vertex. Complete the projections of the solid, as shown in the figure.

The angle of the section situated on the edge ($v\ a$, $v'\ a'$) will be the point (o, o'), in which that edge meets ($v\ r$, $t'\ r'$), the intersection of its vertical projecting plane with the plane P Q R. The remaining angular points may be determined in the same manner; and by joining these points the section ($m\ n\ o$, $m'\ n'\ o'$) will be completed (Prob. IX. Chap. II.)

The real magnitude of the section will be found at once by Prob. XXXVIII. Chap. II.

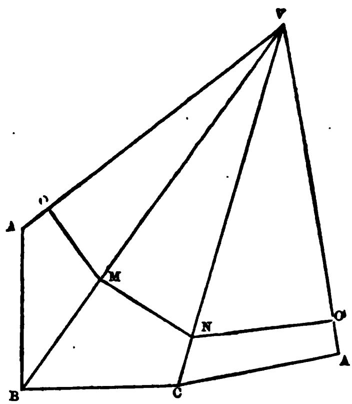
The development will evidently consist of as many triangles as there are sides to the base. These triangles will have their bases equal to the sides of the base of the pyramid; the vertex V (Fig. 90) common, and the sides V A, V B, V C, equal respectively

Fig. 89.



to the edges $(v\,a, v'\,a')$, $(v\,b, v'\,b')$, $(v\,c, v'\,c')$. The lengths of these edges will be found, by turning them about the vertical line $v\,v'$ until they are parallel to the vertical plane of projection, as shown in Fig. 89 to be $v'\,a''$, $v'\,b''$, $v'\,c''$. Make $\mathrm{V\,A} = v'\,a''$; $\mathrm{V\,B} = v'\,b''$; $\mathrm{A\,B} = a\,b$; $\mathrm{B\,C} = b\,c$; $\mathrm{V\,C} = v'\,c''$; $\mathrm{C\,A} = c\,a$; $\mathrm{V\,A} = v'\,a''$. This will be the development of the pyramid Draw $o'\,o''$, $m'\,m''$, $n'\,n''$ parallel to $x\,y$; make $\mathrm{V\,O} = v'\,o''$;

Fig. 90.



V M $= v' m''$; V N $= v' n''$; V O $= v' o''$; join O M, M N, N O. This will be the development of the perimeter of the section.

NOTE.—The section of a right cone may be determined in a similar manner.

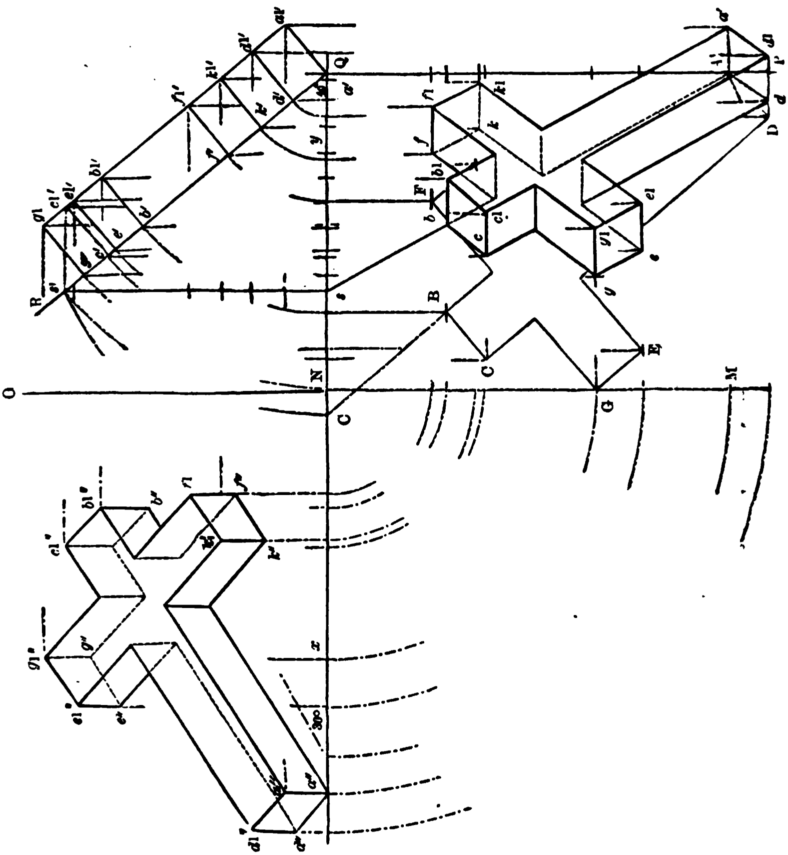
PROBLEM XII.

To draw the plan and elevation of a cross lying on a given plane and inclined at a given angle.

Let P Q R (Fig 91) be the given plane, its horizontal trace being perpendicular to and its vertical trace making an angle of 40,° with $x\,y$; in it place a line inclined at 30°, the plan of the line being A s, and the line itself when turned into horizontal plane being A C; on this make A B equal to the length of the shaft of the cross, and complete the outline A D E G C B F of the cross in its real magnitude. The plans d, e, g, c, b, f may then be found in the usual way: from these points draw perpendiculars to $x\,y$ meeting Q R in d', k', f', b', e', c', g', these will be the elevations: draw $g_1{}'g'$ perpendicular to Q R and equal to the thickness of the cross; draw $g_1{}'\,a_1{}'$, parallel, and $a'a_1{}'$, $d'd_1{}'$, $k'k_1{}'$, $f'f_1{}'$

$b\,b_1'$, $e'e_1'$, $c'c_1'$, perpendicular to Q R; this completes the elevation. The plan may be finished, and the elevation on the

Fig. 91.



third plane, whose traces are M N and N O, be drawn by a process similar to that in Problem VII. Chap. IV.

EXERCISES.

1. One of the faces of a regular octohedron of 2 in. edge is vertical, and an edge of that face inclined at 20°. Draw a plan of the solid, and show upon it each even-numbered contour, making the lowest point zero.

2. A square truncated pyramid, the sides of the base and top measuring respectively $2\frac{1}{2}$ in. and 1 in., and the axis of the trunk 3 in., lies with one of its faces resting on a horizontal plane; project its plan.

3. A solid, similar to that in Ex. 2, lies with its lower face resting on the upper face of the first solid, the plans of the axes of the two solids bisecting each other at right angles. Draw the plan of the second solid, and also an elevation of both on a plane at right angles to the axis of either.

4. Find the intersection of a plane inclined at 40° to the horizon with a cylinder, having for its bases circles of 1 inch in diameter, and standing in a vertical position.

5. A hexagonal right prism, 9 feet long, each edge of the base measuring 2 feet, rests with one face on the horizontal plane. Draw its plan, and an elevation on a plane, making an angle of 27° with the axis of the prism. Scale $\frac{1}{24}$.

6. A tetrahedron, with an edge of 2 inches, rests on one of its faces. Draw its plan, and a sectional elevation on any plane not passing through the apex, and not parallel to an edge of the base.

7. Draw the plan of the frustum of a hexagonal pyramid, the side of the base being $1\frac{1}{2}$ in., the side of the top 1 in., and the height $2\frac{1}{2}$ in., one side of the base being horizontal, and the plane of the base inclined 27° to the horizon.

8. A rectangular parallelopiped, 12 ft. by 10 ft. by 8ft., lies on its base on a plane inclined to the horizon at an angle of 20°, the horizontal trace of the plane making, with the line of level, an angle of 25°. The projection of the longest side of the base (=12 ft.) is inclined to the trace of the plane at an angle of 10°.

Construct the horizontal and vertical projections of the body. Scale $\frac{1}{60}$.

9. The dimensions of a cross are as follows :—

Pedestal $3' \times 3'$ and $1'$ in height; shaft $1' \times 1'$ and $7'$ in height, $1'\ 3''$ of which is clear above the arms; arms $1' \times 1'$ in section, and $1'\ 6''$ long. Draw a front elevation of the cross, its plan, and a section on a line crossing the arms diagonally. Scale $\frac{1}{24}$.

10. Construct a pentagon A B C D E on a side A B equal to 1·3 in.; and considering this the base of a prism 2 in. high, standing on a horizontal plane, draw a vertical section on a line passing through A B.

11. Draw an elevation of the prism in Ex. 10 on a plane, making an angle of 5° with one of the faces.

12. A prism, 3 inches long, having for its ends regular hexagons of 1 inch side, is laid on one of its faces. Draw the plan; and an elevation on a line, making an angle of 30° with the side of the plan.

13. Draw the plan of a pentagonal pyramid, resting upon the plane of its base, each side of the base being 2 inches long, and the height 3 inches. Draw an elevation of the pyramid on a vertical plane parallel to one of the sides of the base.

14. Draw the plan of a frustum of a square pyramid: side of large end 2 inches; side of small end 1 inch; height 3 inches. Draw the elevation on a line making an angle of 40° with one side of the plan of the base.

15. Draw the plan of the prism in Ex. 10, resting on one of the edges of its base: the base being inclined to the horizon at an angle of 15°.

16. Draw the plan of an octohedron, of 2·5 in. edge, lying on one of its faces; and make a section and elevation on any line not parallel to a side of the plan.

17. A prism, 2 inches high, the base of which is a hexagon 1 in. side, stands on a plane inclined at an angle of 40° to the horizon, and has one diagonal of its base inclined at 25° to the horizon. Draw its plan.

18. The face of a cube whose edge measures 2 inches, is inclined 50° to the horizon, and one of the diagonals of this face 25°; project the cube. Draw a horizontal contour $\frac{1}{2}$ inch vertically below the highest point of the cube, and make an elevation of the cube on a vertical plane, parallel to any one of its diagonals.

19. A right prism, each edge of which measures 15 feet, stands with its base, which is a regular pentagon with a side of 8 feet, upon a horizontal plane; draw its plan and also its elevation on a vertical plane making an angle of 20° with any side of the base. Scale 5 feet to an inch.

20. Supposing the prism in Ex. 11 to be oblique instead of right, its axis making an angle of 40° with the plane of the base, draw its plan and elevation as before.

21. A plane is inclined at $46\frac{1}{4}$° to the horizon; on it is placed a tetrahedron of 3 inches edge, and having one side of its base inclined at 25°. Draw the plan and elevation, and contour the former at vertical intervals of ·25 inches.

22. Construct the plan of a hexagonal pyramid resting on one of its faces, its height being 35 feet, and each side of its base 15 feet. Scale 10 feet to an inch.

23. A dodecahedron, formed by two hexagonal pyramids joined at their bases, is laid on a face. Draw its plan; also a sectional elevation on a line cutting the junction of the pyramids at an angle of 45°. Height of pyramid $2\frac{1}{2}$ inches, side of base $1\frac{1}{4}$ inch.

24. Draw a plan, a cross section, and a front elevation of a cottage, 25 feet long, 15 feet wide (interior dimensions), walls 1·5 feet thick, a door in the centre of one side, 4 feet wide and 7 feet high, and a window on each side of it 3 feet wide, 4 feet high, and 4 feet above the ground. The walls of the cottage are 10 feet high, the roof has two gables, and the ridge is 16·5 feet above the floor; the eaves project 9 inches. Scale $\frac{1}{84}$.

25. A cubical block of masonary 20 feet high, having a frontage of 17 feet, and a depth of 13 feet, is perforated longitudinally and transversely by semicircular arches, springing at the same height,

viz. 10 feet 6 inches, and intersecting each other. The piers are, in plan, 4 feet 3 inches square.

Draw the plan of the structure, its front elevation, and a transverse section through the corner of the arch. Scale $\frac{1}{96}$.

26. Construct the plan of a cube 2·5 inches side, having two of its adjacent edges inclined 30° and 40° to the horizon. Show the intersection with the cube of a horizontal plane 1 inch below its highest point.

27. An octagonal right prism, (length 5·5″, diameter of base 3·5″,) stands on one end, and is cut by a plane inclined at 40°, the horizontal trace being parallel to one side of the plan, and 2″ from it; show the true form of the section.

28. Draw the plan of the lower frustum when standing on the section above determined.

29. Draw the plan and elevation of a cube of 2·5″ edge in any position, provided no edge is either horizontal or vertical.

30. The plane containing two diagonals of an octohedron, (edge 3″,) is inclined at 50°, and one of these diagonals is inclined at 37°. Draw the plan of the solid, and an elevation on a plane parallel to the second diagonal.

31. A square prism 4 in. long, side of base 2·5 in., has the three angles A, B, C of one end 5, 10, 25 units high respectively. Draw the plan of the solid. (Unit =·1 inch.)

32. Determine the vertical section of the prism in Ex. 31, by a plane passing through its highest point and parallel to the horizontal trace of the base.

33. Draw the projections of a cube of 3 in. edge, when two of its adjacent faces are inclined at angles of 45° and 60° to the H. P. What is the inclination of the third face of the cube?

34. A hexagonal pyramid, edge of base 1·6 in., axis 3·6 in., rests on one edge, and has the two adjacent faces equally inclined. Draw its plan, and a section on a vertical plane passing through its highest point and making an angle of 30° with the plan of the axis.

35. Draw any three lines meeting in a point and containing obtuse angles: figure them so that they shall represent a solid right angle.

36. The plans *a b*, *a c*, *a d* of three indefinite lines contain angles of 110°, 120°, 130°. A B is inclined at 30°; determine the inclinations of A C and A D if they are at right angles to each other and to A B.

37. A cube, side 20 units, rests on an edge on the H. P.; a face containing this edge is inclined at 40°. Draw its plan and elevation.

38. Find the sectional elevation of a right cone, height 30 units, radius of base 5 units, which, standing on the horizontal plane, is cut by a vertical plane passing through a point 3 units distant from the centre of the base.

39. A pyramid 28 units high, on a square base of 9 units side, stands on a plane inclined at 35° to the horizon, one edge of the base being inclined at 25°. Draw its plan and elevation.

40. The plan, *a b*, of the diameter of a sphere, is 3 in. long, the point *a* is figured 25; *b*, 50. Draw the plan of the sphere, and determine a plane touching it in the point B.

41. The sides and ends of a barge, the plan of the top of which is rectangular, slope $\frac{2}{1}$ and $\frac{4}{1}$ respectively. Determine the angle which they form with each other.

42. A B C is an equilateral triangle of 3 in. side, V a point within it, 1·25 in. from A; 2·5 in. from C. This is the plan of a pyramid standing on A B C; V being figured 38, determine the true form of the face V A C, and the inclination of the face V B C.

43. A regular pentagon, of 1·5 in. side, has two adjacent sides inclined at 20° and 37°: draw the plan of the pyramid of which this is the base, the axis being 3·5 in. long. Draw an elevation, the ground line being parallel to the shortest line of the plan.

44. A pyramid, 4 in. high, has a pentagon of 1·5 in. side for its base; represent it—

(1) With an edge vertical. (2) With a face horizontal.
(3) With two adjacent angles of its base and the opposite

one respectively 1 in., 1·75 in., and 3 in. above the horizontal plane.

45. Draw the plan and elevation of a hollow parallelopiped; height 3″, base 2″ by 2·5″, two edges of base inclined at 35° and 45°. Thickness of material $\frac{1}{3}$″.

46. A pyramid, 4 in. high, has for its base an equilateral triangle, A B C, of 3 in. side: the face V A B, is inclined at 65°; V B C at 75°. Draw a plan and elevation, and find the inclination of the other face.

47. A cube of 2·5 in. edge is turned in a lathe, on the three axes passing through the centres of opposite faces in succession. Show the solid remaining by a plan and two elevations; one of them on a plane not parallel to a face of the cube.

48. A right prism, the section of which is an equilateral triangle of 2 in. side, is 3 in. long. Draw the plan of the prism when one of its faces is inclined at 40°, and an edge in that face at 25°.

49. The diagonals of a cube are 3·5 in. long, and one of them is vertical. Draw a plan of the cube, affixing the proper indices to its angles, the lowest one being zero. Determine the inclination of the three under faces.

50. The base of a right pyramid 4 in. high is a regular pentagon of $1\frac{1}{2}$ in. edge. Draw a plan of the solid when the adjacent edges of the base and one face are inclined at 50° and 30° respectively. Draw also an elevation of the pyramid on a plane parallel to the edge inclined at 30°.

CHAPTER V.

ON SHADOWS.

1. Every deprivation of direct light produces on the surface of bodies an obscurity, more or less intense, which is termed a SHADOW. The Theory of Shadows is based upon the principle that light proceeds in right lines. The determination of shadows comprises two entirely distinct parts; one of these consists in finding the outlines of shadows, the other relates to the depth of the tint to be assigned to each particular portion of the surfaces that receive the shadows. The former of these, as an application of Descriptive Geometry, will alone be treated of here.

2. A RAY OF LIGHT is the term applied to that portion of light which may be looked upon as coincident with a straight line drawn from any point of the luminous body to a point of the object illuminated.

3. If the luminous body be at a very great distance from the body illuminated, as, for example, the sun from the earth, the rays of light may, for all practical purposes, be regarded as parallel to one another. In the present chapter, all objects will be supposed to be illuminated by direct solar light. The direction of the rays will be given by their inclination to one or more given planes.

4. Shadows are divided into two kinds, SHADOWS PROPER and SHADOWS CAST. A SHADOW PROPER is that which takes place on that portion of the opaque body which is turned away from the light. A SHADOW CAST is that which is produced on a surface by the opaque body intercepting those rays of light which would otherwise illuminate that surface.

5. THE LINE OF SEPARATION OF LIGHT AND SHADOW is the line which separates the illuminated portion of a body from that

which is not so; it is determined by the contact of luminous rays with the surface of the body, and is therefore the apparent outline of a body to a spectator whose eye is situated at a point infinitely distant from the body, on a straight line drawn from the body, parallel to the direction of the luminous rays.

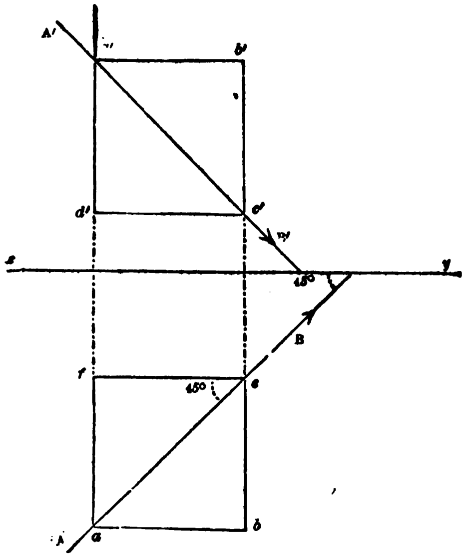
To determine the outline of the shadow cast by an opaque body upon any surface, it will be necessary to find the points in which the rays tangential to the body meet the given surface; the line joining these points will be the outline required.

6. The line of separation on a polyhedron is composed of those edges of the solid which are common to two faces, one of which is struck by the rays of light, the other not. For the face of a polyhedron is either wholly illuminated, or when other faces are interposed between its plane and the source of light, wholly obscure. It will therefore be evident that the line of separation on a polyhedron must be composed of those edges which separate the light faces from the dark ones. The question of determining this line thus depends upon finding those edges. Now, two adjacent edges of a polyhedron will manifestly be, one illuminated, the other not, when the plane drawn through their common edge, parallel to the direction of the luminous rays, leaves both of these faces on the same side of it; for the rays will reach the face in front, but will not reach the face behind, because they are prevented from so doing by the first face and others contiguous to it. If, on the contrary, the plane parallel to the rays of light enters the dihedral angle of two faces, these faces will be both illuminated when the angle is turned towards the luminous source; both deprived of light when the angle is turned away from that source. It will easily be perceived that the plane drawn through an edge parallel to the rays of light, leaves on one side the two faces to which that edge is common, when the trace of that plane on the plane of projection is exterior to the angle formed by the traces of these planes on the same plane.

In the following problems, when nothing is stated to the contrary, the direction of the rays of light will be assumed parallel to the diagonal of a cube whose opposite faces are parallel to the planes of projection. The projections of the rays will then make angles at 45° with the ground line.

Let $a\,b\,e\,f$ (Fig. 92) be the plan, $a'\,b'\,c'\,d'$ the elevation of a cube so situated; then A′ B′ will be the elevation, and A B the

Fig. 92.



plan of a ray of light coincident with the diagonal; and it is clear that A B and A′ B′ are inclined at angles of 45° to $x\,y$.

Problem A.

To determine the shadow cast by a physical point upon the vertical plane of projection.

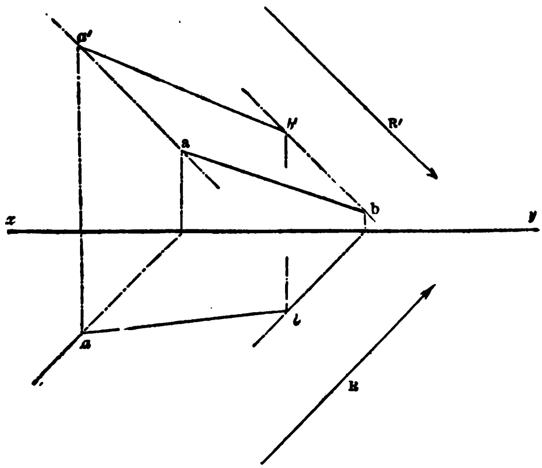
Let R and R′ (Fig. 93) be the projections of a ray of light, (a, a') the given point.

Then, from what has been stated in the introductory remarks, it will be perceived that the shadow cast by the point upon the vertical plane will be the vertical trace of the ray of light passing through the point.

Through (a, a') draw a straight line parallel to (R, R′) (Chap. II. Prob. II.); find the vertical trace, a, of this line (Chap. II. Prob. III.); the point a will be the shadow required.

Similarly, the shadow cast upon the horizontal plane may be found.

Fig. 93.



Cor. Let $(a\ b, a'\ b')$ be a straight line; then, since the shadow of a straight line upon a plane is a straight line, if the shadows of (a, a') and $(b\ b')$, determined as above, be a and b, the straight line a b will be the shadow cast by the line $(a\ b, a'\ b')$ upon the vertical plane.

Its shadow upon the horizontal plane may be determined in a similar manner.

NOTE.—If a straight line be parallel to a plane, its shadow upon that plane will be equal and parallel to the line itself. (*Euc.* I. 33.)

PROBLEM B.

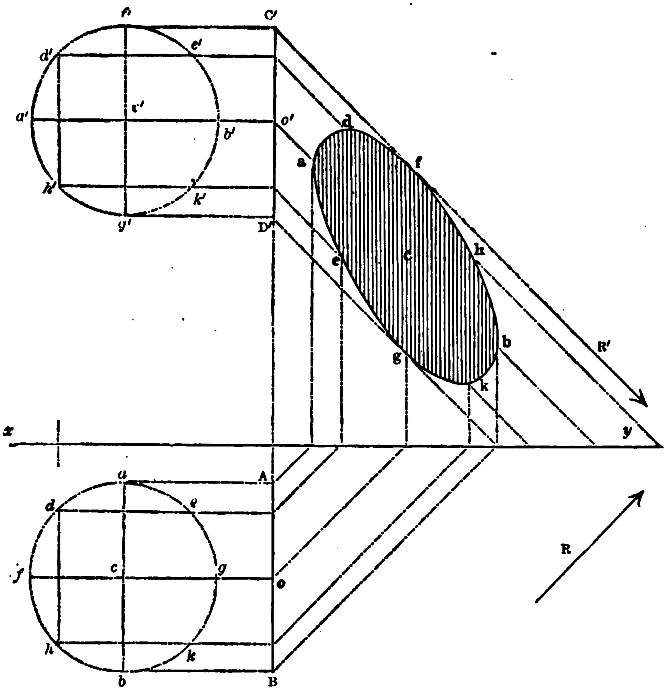
To determine the shadow cast by a circle upon the vertical plane of projection.

(1.) Let the plane of the circle be parallel to the vertical plane the shadow will then be a circle equal to the original

one. Find the shadow of the centre by Prob. A: with this point as a centre, and a radius equal to that of the given circle, describe a circle; this will be the shadow required.

(2.) Let the circle be perpendicular to the planes of projection; its projections (which will be straight lines equal to its diameter,

Fig. 94.



and perpendicular to x y) being A B and C′ D′. Describe the circle in both planes as $a\ f\ b\ g$ and $a'\ f'\ b'\ g'$; divide these circles into equal parts and project the points of division upon A B and C′ D′ respectively. Let the shadows of these points found by Prob. A be, a, d f, h, b, k, g, e, then the curve a d f h b k g e will be the shadow sought, c being the shadow of the centre.

Problem C.

To determine the line of separation on a hexagonal pyramid and the shadows cast by it on the planes of projection.

Let (v, $a\ b\ c\ d\ e\ f$, v', $a'\ b'\ c'\ d'\ e'\ f'$), (Pl. II. Fig. 1) be the pyramid standing on its base on the horizontal plane, (v, v') being the vertex.

It is evident that the faces ($v\,a\,f$, $v'\,a'\,f'$), ($v\,e\,f$, $v'\,e'\,f'$) and ($v\,e\,d$, $v'\,e'\,d'$) are the only ones that are illuminated; ($v\,a$, $v'\,a'$) and ($v\,d$, $v'\,d'$) will therefore constitute the line of separation; the elevation $v'\,a'$ not being visible. The problem will be solved by tracing the shadow of the lines ($a\,v$, $a'\,v'$) and ($d\,v$, $d'\,v'$). To effect this find (Prob. A) V, the shadow of the vertex on the horizontal plane: join a V, d V; this will give the outline of the shadow cast upon the horizontal plane. If the pyramid be sufficiently near the vertical plane, which will be shown by the point V falling above $x\,y$, a portion of the shadow will fall upon that plane. When such is the case determine V′, the shadow of the vertex on the vertical plane (Prob. A): let a V and d V cut $x\,y$ in $\underline{m}$ and $\underline{n}$ respectively: join V′$\underline{m}$ and V′$\underline{n}$; V′ $\underline{m}$ $\underline{n}$ will be the portion of the shadow falling upon the vertical plane.

(Fig. 2, Pl. II.) shows the shadow of a hexagonal prism, obtained in a similar manner, and consequently needing no explanation.

Problem D.

To determine the line of separation on a right cylinder with circular base, and the shadow cast by it on the planes of projection.

Let the cylinder stand upon its base upon the horizontal plane. The determination of the line of separation reduces itself to drawing two tangents to the base parallel to the direction of the light, and to drawing through the points of contact two generatrices of the cylinder

Let $a\,b\,c\,d\,e$ and $a'\,c'\,c''\,a''$ be the plan and elevation of the solid (Pl. II. Fig. 3), draw $e\,\underline{m}$ and $b\,\underline{n}$ parallel to R the plan of a ray of light and touching the circle $a\,b\,c\,d$, in b and e; draw $e\,e''$ and $b\,d''$ perpendicular to $x\,y$; $e'\,e''$ and $d'\,d''$ will be the line of separation, $e'\,e''$ being invisible. Let $e\,\underline{m}$ and $b\,\underline{n}$ meet $x\,y$ in $\underline{m}$ and $\underline{n}$; find (Prob. A) $\underline{e}$, $\underline{f}'$, $\underline{d}'$, $\underline{c}'$, $\underline{g}'$, the shadows of points, in the upper end, on the vertical plane. The curve $\underline{e}'\,\underline{f}'\,\underline{d}'\,\underline{c}'\,\underline{g}'$ and the lines $e\,\underline{m}$, $\underline{g}'\,\underline{n}$ will complete the shadow thrown on that plane. Had the shadow fallen wholly on the horizontal plane, it would have been determined by describing a circle with the shadow of the centre for a centre, and touching $h\,\underline{m}$ and $h\,\underline{n}$, as shown in the diagram.

Problem E.

To determine the line of separation on a right cone, and the shadows cast by it on the planes of projection.

1. Let the cone (v, $a\,b\,c\,d\,e$, v', $a'\,c'$), (Pl. II. Fig. 4), stand on its base upon the horizontal plane. Determine, by Prob. A, v the shadow of the vertex; from v draw v b and v e, touching the base in b and e; this will give the outline of the shadow cast upon the horizontal plane. Join the points of contact e and b with v; the lines ($v\,e$, $v'\,e'$) and ($v\,b$, $v'\,b'$) will be the line of separation, $v'\,b'$ alone being visible in the elevation.

When a portion of the shadow falls upon the vertical plane it may be determined as in Problem C.

2. Let the cone be inverted, v, $a\,b\,c\,e$ and $a'\,v'\,d'$ (Pl. II. Fig. 5) being its projections. Determine $\underline{o}$ the shadow of the centre (Prob. A); with $\underline{o}$ as a centre and a radius equal to that of the base of the cone, describe the circle $\underline{m}\,\underline{p}\,\underline{n}$; from o draw $o\,\underline{m}$ and $o\,\underline{n}$, touching this circle in $\underline{m}$ and $\underline{n}$; this will complete the outline of the shadow cast upon the horizontal plane. Again from o draw $o\,b$ and $o\,e$ parallel to $\underline{o}\,\underline{m}$ and $\underline{o}\,\underline{n}$; the lines ($v\,e$, $v'\,e'$) and ($v\,b$, $v'\,b'$) will form the line of separation; the elevation $v'\,b'$ alone being visible.

When a portion of the shadow falls upon the vertical plane, that part may be determined by finding the shadows of equi-

distant points in the base and tracing an elliptical curve through them, in a manner similar to that in Problem B.

Problem F.

NUMERICAL EXAMPLE.

To draw the shadow cast upon the horizontal plane by a solid formed of two pentagonal pyramids joined at their bases. The rays of light making an angle of 50° with the horizontal, and of 15° with the vertical, planes of projection; edge of base ·895 inch; perpendicular height of each pyramid 1·25 inches ; one edge of base making an angle of 34° 30′ with vertical plane.

Let v_1, $a\ b\ c\ d\ e$ (Fig. 6, Pl. II.) be the plan, and $v'_1\ a'\ v'_2\ c'$ the elevation of the solid determined as shown in Chap. IV.

The line of separation found by the principle enunciated in Chap. V. 5, will consist of the lines $(v_1\ e, v'_1\ e')$; $(v_1\ b, v'_1\ b')$, $(v_2\ e, v'_2 e')$, and $(v_2\ b, v'_2\ b')$. Through (v_2, v'_2) draw a straight line making an angle of 50° with the horizontal plane, and an angle of 15° with the vertical (Chap. II. Prob. XV.): find the horizontal trace of this line (Chap. II. Prob. III.) : let it be v_2; v_2 will be the shadow of the point (v_2, v'_2) (Chap. V. 5). The shadows $\underline{e}$ and $\underline{b}$ of the points (e, e') and (b, b') will be found in the same way: join $\underline{e}\,\mathrm{v}_2$, $\underline{b}\,\mathrm{v}_2$; these lines will be shadows of the lines $(v_2\ e, v'_2\ e')$ and $(v_2\ b, v'_2 b')$; the shadows of $(v_1\ e, v'_1\ e')$ and $(v_1\ b, v'_1 b')$ will be found by joining $v_1\ \underline{e}$ and $v_1\ \underline{b}$, because the point $(v_1\ v'_1)$ is in the horizontal plane.

Exercises.

1. Draw the elevation of a cube of $2\frac{1}{2}$ inches edge, lying on one of its faces, on a plane which makes an angle of 25° with another face of the cube : project also the shadow thrown by the cube on the plane on which it lies, the rays of light being parallel to the plane on which the elevation is taken, and making angles of 50° with the horizon.

2. Follow the instructions given in the preceding question, the cube now resting on one of its edges, and a face containing that edge being inclined 25° to the horizon.

3. Project the shadow which the frustum in Ex. 7, Chap. IV., will throw upon the horizontal plane passing through the lower edge of the base, the plan of the rays being parallel to that edge, and inclined 37° to the horizon.

4. A pyramid with a pentagonal base, B C D E F, of $1\frac{1}{2}$-inch side, resting on a horizontal plane, has its vertex A 2 inches perpendicularly over a point within the base, ·88 inch from the angles at B and C. Draw the plan, and a section on a line joining the angles B and D.

5. Supposing the rays of light to be parallel, and to form with the horizon an angle of 65°, give the lines of the shadow on the horizontal plane on which the pyramid in Ex. 4 rests, when one of the edges of the base is inclined to the rays of light at an angle of 45°.

6. Project the shadow which the oblique prism in Ex. 20, Chap. IV., would cast upon the horizontal plane upon which it stands, the rays of light making an angle of 50° with the horizon, and being parallel to the axis of the prism.

7. A square pyramid, of 2 inches edge, is placed with the vertex downwards and raised 1 inch above the paper. Draw its plan and its shadow when the rays are inclined at an angle of 30° to the horizon.

8. Draw the plan of an octohedron, with the line joining the vertices in a vertical position; and the elevation on a vertical plane, the trace of which makes an angle of 35° with a side of the plan of the octohedron. Also the shadow thrown by a ray, making an angle of 40° with the horizon and 36° with side of plan. Put a flat tint on the shadow and on the faces in shade, darkening those most shaded. Edge of solid 2 inches.

9. Draw the plan of a tetrahedron, the edge being $2\frac{1}{2}$ inches, and a sectional elevation of the solid on any vertical plane not passing through the vertex. Determine the shadow thrown by the solid on the plane of its base, the rays of light being parallel to one edge of the base and inclined at an angle of 30° to the horizon.

10. Project the shadow that would be thrown by the pyramid

in Ex. 22, Chap. IV., upon the horizontal plane upon which it rests, the parallel rays of light making an angle of 54° with the horizon, and a horizontal angle of 36° with the axis of the pyramid.

11. Draw the shadow which would be cast by the frustum, Ex. 2, Chap. IV., on the horizontal plane, when the sun's rays make with the horizon an angle of 49°, their plans being parallel to the diagonal of the base of the solid.

12. Draw the shadow which the prism in Ex. 10, Chap. IV., would throw on the plane on which it stands, supposing the rays of light to fall on one of the faces in a direction inclined at 45° both with its vertical and with its horizontal boundary.

13. Project the shadow that would be thrown by the pyramid upon the horizontal plane upon which it stands, the parallel rays of light making an angle of 40° with the horizon. Ex. 13, Ch. IV.

14. Draw the plan of a cylinder 3 inches long resting on its side, and project the shadow when the ray of light makes an angle of 40° with the horizon, and its plan an angle of 30° with the side of the plan of the cylinder. Diameter of end 2 inches.

CHAPTER VI.

ISOMETRIC PROJECTION.

THIS method of projection, based upon that of a cube situated with its diagonal perpendicular to the plane of projection, affords a means of representing objects in a manner somewhat resembling what is called a "bird's-eye view." Its adaptability to this purpose was first pointed out by Professor Farish, of Cambridge.

The term isometrical is applied to it because the projections of all straight lines parallel to any edge of the cube may be measured from the same scale. This is evident because such lines are all inclined at the same angle to the plane of projection; and, consequently, the projections of any two lines will have the same ratio as the lines themselves.

This kind of projection is peculiarly suitable to the delineation of objects whose bounding surfaces lie in three planes which form a right trihedral angle.

PROBLEM.

To determine the projection of a cube with its diagonal perpendicular to the plane of projection.

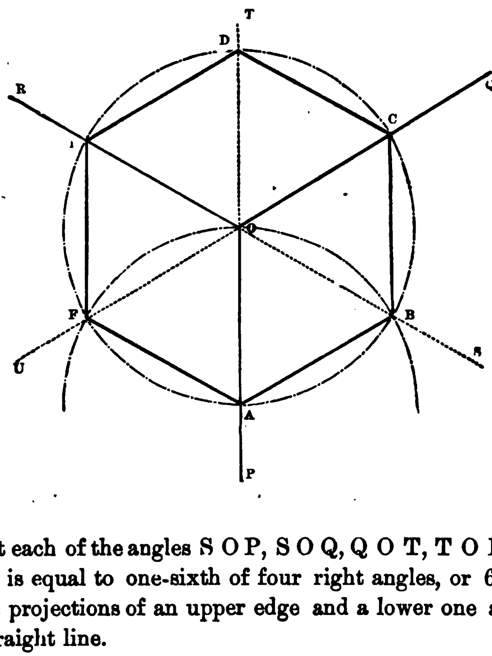
Let O (Fig. 95) be the projection of the diagonal which is perpendicular to the plane of projection.

Then the three edges which meet the lower extremity of this diagonal will be projected in three straight lines passing through O, and making angles of 120° with one another. For these lines make equal angles with one another, and are inclined at the same angle to the plane of projection. Let these projections be O S, O T, O U.

To obtain the projections of the three edges coterminous with the upper extremity of the diagonal, it must be observed: (1) that O is a point in each of them; (2) that their projecting planes are the same as those of O S, O T, O U; these projections will, therefore, be found by producing S O, T O, U O, as O R, O P, O Q.

It is thus seen that the six edges of the cube coterminous with the diagonal are projected in six straight lines radiating from O;

Fig. 95.



and that each of the angles S O P, S O Q, Q O T, T O R, R O U, U O P, is equal to one-sixth of four right angles, or 60°: also, that the projections of an upper edge and a lower one are in the same straight line.

It remains to determine the magnitude of these projections, which will evidently be equal to one another.

Let *a b* (Fig. 96) be the edge of the cube: draw *b c* at right angles to *a b* and equal to it: join *a c*: draw *c d* at right angles to *a c*, and equal to *a b*: *a d* will be the diagonal of the cube: complete the rectangle *a c d e*: this will be the diagonal plane of

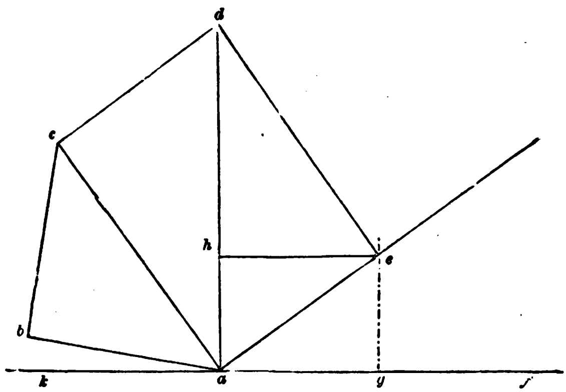
the cube; and will be at right angles to the plane of projection when $a\ d$ is. (*Euc.* XI. 19.)

Draw $k\ f$ through a, perpendicular to $a\ d$: draw $e\ g$ perpendicular to $k\ f$: $a\ g$ will manifestly be the projection of $a\ e$.

Referring again to Fig. 95, with centre O and radius O A, equal to $a\ g$, describe a circle meeting the projections in A, B, C, D, E, F: O A, O B, O C, O D, O E, O F will be the projections of the six edges which meet the diagonal whose projection is O.

The projections of the remaining six edges must be the lines

Fig. 96.



A B, B C, C D, D E, E F, F A; which are equal to one another and to O A, because the triangles A O B, B O C, &c., are equilateral.

Cor. 1. The outline of the projection of a cube is a regular hexagon inscribed in the circle whose radius is equal to the projection of an edge.

Cor. 2. The projection of each face is a rhombus, composed of two equilateral triangles; consequently its angles are 120°, 120°, and 60°, 60°.

Cor. 3. The ratio between a straight line and its isometric projection may be thus determined. See Fig. 96. Let $n =$ the

edge of the cube, draw $e\,h$ perpendicular to $a\,d$. Then since $a\,b = b\,c = n$; $a\,c = n\sqrt{2}$, and since $c\,d = n$; $a\,d = n\sqrt{3}$. (*Euc.* I. 47.)

Again (*Euc.* VI. 8):

$$a\,d : d\,e :: d\,e : d\,h, \quad \therefore d\,h = \frac{2n}{\sqrt{3}}$$

also, $a\,d : a\,e :: a\,e : a\,h, \quad \therefore a\,h = \dfrac{n}{\sqrt{3}}$

and, $d\,h : e\,h :: e\,h : a\,h, \quad \therefore e\,h = \sqrt{\dfrac{2\,n}{\sqrt{3}} \times \dfrac{n}{\sqrt{3}}} = \dfrac{n\sqrt{2}}{\sqrt{3}}.$

but $a\,g = e\,h, \quad \therefore a\,g = \dfrac{n\sqrt{2}}{\sqrt{3}} = \dfrac{n}{3}\sqrt{6} = \cdot 8165\,n$, nearly.

The required ratio is, therefore, $3 : \sqrt{6}$ or $1 : \cdot 8165$, nearly.

Cor. 4. If θ be the inclination of the edge,

$$\text{Cos}\,\theta = \frac{a\,g}{a\,e} = \cdot 8165, \therefore \theta = 35^\circ\,16'.$$

If ϕ be the inclination of a face,

$$\text{Sin}\,\phi = \text{Cos}\,\theta = \cdot 8165, \therefore \varphi = 54^\circ\,44'.$$

Def.—The lines O P, O Q, O R, are called axes; and lines can be projected isometrically only when they are parallel to some one of the lines projected in O P, O Q, O R.

EXAMPLE 1. To construct the isometrical projection of a scale of $\frac{1}{12}$.

Let $a\,e\,d$ (Pl. II. Fig. 7) be a right-angled triangle, constructed as in Fig. 96, the side $a\,e$ being 1 inch; so that $a\,d$ is the diagonal of a cube of 1 inch edge; draw $k\,f$ perpendicular to $a\,d$; in $a\,e$ produced set off $e\,m = m\,n = a\,e = 1$ inch: draw $n\,f$, $m\,g$, $e\,l$ perpendicular to $k\,f$: $k\,l$, $l\,g$, $g\,f$ will clearly be the isometric projections of straight lines each one inch long. Divide $a\,e$ into twelve equal parts, and draw through the points of subdivision straight lines parallel to $a\,d$: the line $k\,f$ will thus be graduated so as to form a scale of $\frac{1}{12}$ if used for feet; or of $\frac{1}{1}$ if used for inches.

Ex. 2. To construct an isometric scale of $\frac{1}{50}$ to measure feet. Let the scale represent 10 feet.

Then $50 : 10 :: \overset{\text{in.}}{12}$: No of inches in the length of scale.

$$\therefore \text{ length} = \frac{\overset{\text{in.}}{12} \times 10}{50} = \overset{\text{in.}}{2 \cdot 4}$$

Draw a line L (Pl. II. Fig. 8) 2·4 inches long: divide it into five equal parts, each will show 2 feet: subdivide the first primary division into 4 equal parts, each will show $\frac{1}{2}$ foot. The line L is, therefore, a scale of $\frac{1}{50}$.

To find the isometric projection of this scale: draw a line M making with L an angle equal to the angle which *a e* makes with *k f* (Fig. 7), and proceed as in Ex. 1. These scales should be completed as shown in the diagrams; their use will be exemplified in the following examples.

Ex. 3. To draw the isometrical projection of a rectangular block of wood: out of which a circle has been cut, the centre of the circle coinciding with the intersection of the diagonals of one face of the block.

Dimensions of block, $\overset{\text{ft.}}{10} \times \overset{\text{ft.}}{10} \times \overset{\text{ft.}}{2\frac{1}{2}}$. Radius of circle, $\overset{\text{ft.}}{3}$. Scale, $\frac{1}{50}$.

Let *e h*, *e a*, *e f* (Pl. II., Fig. 9) be the axes. Set off on these axes *e h* and *e f*, each equal to 10 feet measured on the scale in Fig. 8: complete the parallelogram *e f g h*; this will be the projection of a face of the solid $\overset{\text{ft.}}{10} \times \overset{\text{ft.}}{10}$: make *e a* equal to $2\frac{1}{2}$ ft.; complete the parallelograms *e a b f*, *e a d h*: these will be the projections of two faces each $\overset{\text{ft.}}{10} \times \overset{\text{ft.}}{2\frac{1}{2}}$; and the projection is completed. The dotted lines *b c*, *d c*, *c g* are the projections of edges not visible.

To draw the projection of the circular aperture.

Let the diagonals *h f* and *g e* intersect in *x*: set off *x m* equal to *x k*, equal to 3 feet, taken from the same scale as before: complete the rhombus *k l m n*: this will be the projection of a square of 6 feet side: it is required to construct the projection of the circle inscribed in this square, which will be an ellipse inscribed in the rhombus *k l m n*.

It is evident that *o*, *p*, *q*, *r*, the middle points of *n k*, *k l*, *l m*, *m n*, will be four points in this curve: its major axis will be situated in *h f*, and, being the projection of a straight line parallel to the plane of projection, will be equal to the diameter of the circle. If, therefore, *x s* be made equal to *x t*, equal to 3 feet taken from the line L in the scale Fig. 8: *s* and *t* will be the extremities of the axis major.

The minor axis bisecting the major at right angles must coincide with *e g*: its extremities, *u* and *v*, will be determined by drawing *t u*, *s u* parallel to *h g*, *f g*, and meeting *e g* in *u*; and *t v*, *s v* parallel to *h e*, *f e*, meeting *e g* in *v*: the figure *v s t u* being the projection of a square inscribed in the circle. Eight points in the ellipse being thus determined, the curve can be traced through them.

The projection of the circle in the face *a d m b* may be obtained by a similar construction.

Fig. 10 is the isometric projection of a wooden tray 1 ft. 8 in. long, 11¾ in. broad, 5¾ in. deep, of material 1⅓ in. thick; on a scale of $\frac{1}{12}$.

Fig. 11 is the isometric projection of a cylinder: height 7·5 feet, radius of base 2 feet. Scale $\frac{1}{50}$.

These have been inserted as examples of the applicability of this kind of projection, though it has not been deemed necessary to append any explanation of the construction.

Exercises.

1. Give the ratio of a straight line to its isometric projection, and prove its correctness by a diagram.

Draw the isometrical projections of the following objects:—

2. A rectangular parallelopiped 12 ft. × 10 ft. × 8 ft. Scale $\frac{1}{50}$.

3. A flight of 10 steps, each 8 ft. long, 1 ft. wide, and 8 in. deep. Scale $\frac{1}{24}$.

4. A table 4 ft. long, 2½ ft. wide, and 2½ ft. high, the top 2 inches thick, the legs 2 in. square, and fixed 1½ in. from the

outside of the table, with a circular hole $1\frac{1}{2}$ ft. in diameter in the middle of the top. Scale $\frac{1}{12}$.

5. A hexagonal right prism, height 9 ft., side of base 2 ft. Scale $\frac{1}{24}$.

6. A table 4 ft. long, $2\frac{1}{2}$ ft wide, and 3 ft high, top 3 in. thick, legs 2 in. square. Scale $\frac{1}{12}$.

7. A circle $2\frac{1}{2}$ in. in diameter.

8. A cylindrical box, made of $\frac{1}{2}$ in. deal, the exterior diameter being 8 inches and the height 3 inches. Scale $\frac{1}{2}$.

9. A rectangular tray, 3 in. long, 2 in. wide, 1 in. high, and the sides $\frac{1}{4}$ in. thick.

10. A box 2 ft. square, and $1\frac{1}{2}$ ft. deep, made of $\frac{1}{2}$ in. board and having a circular hole 4 in. in diameter in each side and end. Scale $\frac{1}{4}$.

11. A piece of timber, 3 ft. long, $1\frac{1}{2}$ ft. wide, 3 in. thick; a hole, in the shape of the frustum of a cone, is bored through the thickness in the centre of the length and breadth. The upper diameter of the hole is 1 ft., the lower diameter 4 in. Scale $\frac{1}{6}$.

12. The walls of a cottage, from which the roof has been removed, consisting of one room, of which the external dimensions are, length 15 ft., breadth 12 ft., height 10 ft. 6 in. In one of the long sides are two windows with semicircular heads, each 5 ft. 6 in. high to the springing of the arch, 3 ft. 6 in. wide, 2 ft. 6 in. from the ground. In one of the shorter sides is a doorway 3 ft. wide, 6 ft. 6 in. high, reached by two steps, each 3 ft. long, 1 ft. wide, and 6 in. high. The walls are 1 ft. thick. Scale $\frac{1}{48}$.

13. Draw the isometrical projection of a clock face 2·5 feet in diameter, showing in correct drawing the twelve hours in Roman numerals 4 inches long and half an inch from the rim of the dial. Scale $\frac{1}{6}$.

14. Draw in isometrical projection a building ready to receive its roof, of the following dimensions: length 30 ft., breadth 20 ft., height to eaves 10 ft., thickness of walls 2 ft. The end walls are carried up as gables at 45°. At one end show a doorway 7 ft. by 3 ft., with flat stone lintel, having in front of it a step 2 ft.

wide, 6 inches deep, and projecting 6 inches each side of doorway; also three triangular buttresses 2 ft. thick, running the whole height of the side wall, with bases of 3 feet; the buttresses are equidistant from each other. Scale $\frac{1}{72}$.

15. The plan of a certain solid is a circle of 2 inches radius; its front elevation is a square of 4 inches; its side elevation (i.e. on a plane perpendicular to the former) is an isosceles triangle, of 4 inches base and height. Draw the isometrical projection of this solid. Scale $\frac{1}{4}$.

CHAPTER VII.

MISCELLANEOUS EXERCISES.

1. Draw a line inclined at 30° and making an angle of 45° with the vertical plane; and a plane inclined at 50° and making an angle of 65° with the vertical plane, to contain the line.

2. The horizontal trace of a plane makes an angle of 30° with the ground line; draw the vertical trace on the supposition that the two traces really contain an angle of 65°; thence determine the angles which this plane makes with both planes of projection.

3. Draw a line A B, one inch long and inclined at 40°; through A, draw a line perpendicular to A B, and inclined at 20°, and through B a line, also perpendicular to A B, but inclined at 30°.

4. Draw the plan and elevation of an equilateral triangle A B C of 2·5 inches side, when its plane is inclined at 60°, and the side A B at 35°; determine the inclinations of the sides A C, B C.

5. Draw the figured plan and elevation of a square A B C D of 2·5 inches side, (1) when the corners A, B, C, are at 1·1, 2, 2·9 inches above the horizontal plane; (2) when its diagonal B C is inclined at 15°, and its diagonal A D at 38°.

6. Draw a regular pentagon of 1·5 inches side, the plane of which is inclined at 50°, when one diagonal is horizontal, and show its elevation on a plane parallel to another diagonal, and determine the angle which the plane of the pentagon makes with the vertical plane of the elevation.

7. Show, by its plan and elevation, an octohedron of 3·25 inches edge, when the edge A B is horizontal, and the face A B C inclined at 20°.

8. Three spheres of 1·5, 1, ·5 inches radii lie on a horizontal plane, each sphere touching the other two; represent them by their plans, and determine a plane touching all three.

9. The lateral planes of a parallelopiped are given by their traces; the base is in the horizontal plane; the height of the solid is 35 units: draw its plan and elevation. Unit ·1 inch.

10. Draw the plan of a tetrahedron of 4 inches edge, when two sides of the base are inclined 5° and 19° respectively.

11. Draw the plan of a right pyramid with a square base, upon the plane of the latter: the faces are equilateral triangles with sides 15 feet each. Scale $4\frac{1}{2}$ feet to an inch. Add an elevation, and a shadow, the inclination of the rays being 41°.

12. Draw the plan and shadow (each on the plane of the base) of an oblique heptagonal prism, the axis of which makes an angle of 45° with the base. The height of the solid is 30 feet; the scale $\frac{1}{120}$; and the base is a regular heptagon of 13 feet side. Make a sectional elevation on a plane passing through the upper extremity of the axis of the prism and inclined to the plan of that axis 45°.

13. The lines joining three points, which are respectively 1, $1\frac{1}{2}$, and $2\frac{1}{2}$ inches above a horizontal plane, form a triangle, each side of which measures 2 inches *in plan*; show how you determine the real form of this triangle and the inclination of the plane containing it.

14. Draw the plan of a right pyramid 2·3 inches high, in such a position that one of the diagonals of its base, which is a square of 2 inches side, may be inclined at 13°, whilst the other is horizontal; and draw on it also two horizontal contour-lines at the levels of 1 and 2 inches below the apex.

15. A plane cuts the horizontal plane at an angle of 45°; a second plane, cutting the horizontal plane at an angle of 20°, cuts also the first plane. The horizontal traces of the two planes meet when produced at an angle of 15°. Determine the inclination of the line of intersection of the first and second planes, and the angle which it makes with the horizontal of each plane. Name the scale of units adopted.

16. Draw an indefinite line inclined at 30°, and one plane, containing that line, inclined at 50°, another plane also containing the line, but inclined at 65°; determine the angle contained by these two planes.

17. The plans of two lines contain an angle of 56°; one line A B is inclined at 40°; at what angle is the other line A C inclined, if the angle B A C is half a right angle?

18. Draw the plan and elevation of a cube of 3·25 inches edge, when the planes of its faces A B C D and A C E G are inclined at 55° and 69°.

Or when the edges A B, A C are inclined at 46° and 22°.

Or when the corners A, B, C, are at 11, 20, 29 units above the horizontal plane.

19. Express by its two projections, a point in each region of space and distant 20 units from the vertical, and 8·5 from the horizontal, plane. Also express by their figured plans 4 points each distant 30 units from $x\,y$, two *in* the vertical plane and on opposite sides of the horizontal, and the other two *in* the horizontal plane, but upon opposite sides of the vertical.

20. Draw the traces of two planes inclined 35° and 54° respectively to the horizontal plane, and making a dihedral angle of 110° with each other.

21. If a right prism or cylinder be cut by a plane perpendicular to its base, the section will be a rectangle. Prove this.

22. Construct the plan of a right pyramid which has an axis of 35 feet (scale $\frac{1}{108}$) and a regular nonagon of 10 feet side for its base, upon the plane of one of the faces of the solid. Add an elevation upon a plane inclined 34° to the plane of the axis. Draw also the section by a plane bisecting the axis and parallel to the elevation.

23. Draw the plan of a right prism with hexagonal bases on the plane of one of the latter: the sides of the hexagons are 4 feet each, and the height of the solid is 11 feet, scale $\frac{1}{36}$. Project the shadow on the plan by parallel rays inclined 36°. Make a sectional elevation on a plane bisecting two adjacent faces of the prism.

24. Represent by their figured plans the following lines :—

A B 30	units	long—	inclined	at	30°.
C D 30	„	„	„		60°.
E F 30	„	„	„		90°.

25. Show by their scales of slope three planes :—
One inclined at 30°; one inclined at 50°; the other inclined at 70°.

26. Show the vertical trace of each of these planes on a plane making an angle of 25° with the horizontals of each.

27. Show by its traces a plane inclined at 40°, containing a line inclined at 20° (the line to be shown by its plan and elevation).

28. If a second plane inclined at 60° contains the same line, determine the angle the two planes would contain.

29. Draw the plan and elevation of a prism 30 units long, with a regular pentagon of 10 units side for its base.
Either when one edge of the base is horizontal, and the edges are inclined at 20°; or when one diagonal of the base is horizontal, and the edges are inclined at 50°.

30. Determine the two projections of 2 points, one of which is 12 units behind the vertical plane of projection and 6·5 units above the horizontal; and the other 15 units below the horizontal and 7 units in front of the vertical plane.

31. Determine by their contours, or by their traces, planes inclined 45° and 30° respectively to the horizontal.

32. Draw an equilateral triangle, side 35 units, figure the angles of the plan 12, 47, and 25, and determine the original triangle and the inclination of its sides to the horizontal plane.

33. A truncated pyramid on a square base of 1·5 inches side, and the top surface of 1 inch side, is 2·5 inches high. Draw a section on a line crossing the plan diagonally.

34. Draw the horizontal projection of the truncated pyramid in Ex. 33, when its base is inclined at an angle 25° to the horizon and one of the edges of the base rests upon the horizontal plane.

35. Draw an elevation of the same solid when the edge which rests upon the horizontal plane makes an angle of 30° with the ground line.

36. An hexagonal prism of 1·5 inches side, and 4 inches long, is to be shown by a plan and elevation in one of the following positions:—

a. When a diagonal of the solid is vertical.

b. When a diagonal is horizontal.

c. When two opposite edges of the ends are vertically over each other.

37. If A is the area of any plane figure, and a, a' a'', those of its projections on any three rectangular co-ordinate planes, prove that

$$a^2 + a'^2 + a''^2 = A^2.$$

38. A square, $ABCD$, of 2·5 inches side, lies in a plane inclined at 35°; the diagonal $A\ C$ is inclined at 30°; draw the plan and elevation of the square on a plane parallel to the diagonal $B\ D$.

39. Draw the plan of a regular pentagon $A\ B\ C\ D\ E$, of 1·5 inches side, when the diagonals $A\ D$, $B\ E$ are inclined at 20° and 35°.

40. Draw the plan of a cube of 2·5 inches side, in any oblique position; on each of its six faces place a pyramid having for its base the face of the cube, its altitude being half the edge of the cube.

41. Draw a triangle $A\ B\ C$, its sides being 3, 3·5, 4 inches; determine its centre of gravity V; join $A\ V$, $B\ V$, $C\ V$; figure the point V, 4·5; this is the plan of an irregular pyramid 4·5 inches high, standing on its base; draw an elevation on a ground line at right angles to $B\ C$. Determine the real form of the face $B\ C\ V$. Determine the angles contained by the faces $A\ B\ C$, $V\ B\ C$, and $V\ A\ C$, $V\ B\ C$. Draw the plan of the solid when suspended from its corner A.

42. A diameter $N\ S$ of a sphere of 2 inches radius is inclined at 53°; show it in plan, and draw that of the great circle in which a plane inclined at 70°, containing $N\ S$, cuts the sphere.

43. Supposing the great circle in Ex. 42 to be the meridian of Greenwich, determine the place of a town the longitude of which is 60° E., and latitude 40° N.

44. A tetrahedron of 2″ edge has one of its faces inclined at 35°, and an edge in that face horizontal. Draw its plan and elevation on a plane parallel to the horizontal edge.

45. Draw the circle passing through the given points l_{30}, m_{50}, n_{60}. ($lm = mn = nl = 3''$).

46. Construct the scale of slope of the plane of points *L*, *M*, *N* in Ex. 45.

47. The sides of a triangle are 1·5″, 2″ and 2·5″; the shortest side is inclined at an angle of 20°, and the 2″ side, at 35°. There is also a square on each side. Draw the plan.

48. Bisect the angle *D E F* of the triangle $d_{15}\,e_{25}\,f_{40}$ ($de = 2''$, $ef = 2{\cdot}5''$, $fd = 3''$): show the true form of this angle, and the inclination of its plane.

49. An isosceles triangle has a horizontal base of 3″; the vertical angle is 30°, and the plan of the same 90°: determine the angles of inclination of the sides.

50. A cube whose edge is 2·5″ has:—

(1) one face inclined at an angle of 50° and an edge in that face inclined at an angle of 27°.

(2) adjacent faces inclined at angles of 60° and 55°.

Draw the plans.

51. a_{30} is the centre of a sphere radius 2″; b_{50} and c_{30} are given points: determine a plane through *B* and *C* to touch the given sphere ($ab = bc = ac = 3''$).

52. A cube of 3″ side has one face inclined at 32° and one edge at 28°. Draw the projection of the solid, and contour it for every unit of altitude. Give a section of the figure by a plane passing through a diagonal.

53. A right prism, 4 inches long, with hexagonal bases of 1·7 inches side, has two of its faces horizontal; required its plan,

also its elevation on a plane inclined at 40° to the horizontal edges of the solid.

54. Construct the section of the solid in Ex. 53 by a plane passing through the centre of the prism, and parallel to the given plane of elevation. Also draw the projections of the prism when a side of one of its bases is in the horizontal plane, and the face containing that side is inclined 37°.

55. A cube of 3 inches edge has one edge in the horizontal plane, and the faces containing this edge equally inclined; required—

a. A plan of the cube.

b. A vertical section by a plane having its horizontal trace in the diagonal of the plan.

c. An elevation on a plane parallel to the plane of section.

56. Given a right pyramid, height 100 feet, and base a regular heptagon of 22 feet side, scale $\frac{1}{240}$, required—

a. A plan and elevation of the solid when its axis is vertical.

b. Ditto, ditto, when the pyramid rests with one face on the horizontal plane.

c. A vertical section, by a plane passing through the centre of the horizontal face, and making an angle of 45° with its apothem.

57. The edge of a cube is 32 feet; draw its projections when the diagonal of the solid is inclined at 40°, and an edge adjacent to the same is inclined at 23°. Scale for plan $\frac{1}{120}$.

58. A line 47·5 long, inclined at $\frac{1}{12}$, is one edge of a prism, formed by the following planes: one inclined at $\frac{1}{6}$; the second, also containing the line, inclined $\frac{2}{3}$ in the opposite direction; the third inclined $\frac{1}{14}$ in the same direction as the line, the lowest end of which is 6 units vertically above the third plane, the horizontals of which are at right angles to the plan of the line. A plane, sloping downwards and outwards $\frac{1}{1}$, having its horizontals also at right angles to the line, terminates, at each end, this truncated pyramid. Show the whole by contours for every unit of level. Unit, ·1 inch.

59. The top of a small table is 3 feet long, 2 feet wide and 2″

thick ; it is supported on four legs, 3 feet long and 2″ square, placed 1″ back from the edges at the corners. A shelf beneath rests on rails framed into the legs, 6″ above the floor ; three circular holes 4″ in diameter are cut through the top to receive retorts, filtering glasses, &c. Show it in isometrical projection. Scale, $\frac{1}{6}$.

60. Three forces, P, Q, R, as 3 : 5 : 6, act at a point A ; the angle contained between the directions of P and Q is 50°, between P and R is 70°, and between Q and R is 80°. The three forces do not lie in one plane. Determine by construction the resultant in magnitude and direction.

Examination Papers.

I.

1. Two points, A, B, are A, 1·2″ above the horizontal and 2″ in front of the vertical plane, B, ·5″ below the horizontal plane and 1″ behind the vertical plane. Make a second elevation of these points on a vertical plane from which both points are equidistant 2″, and in front of it.

2. A line 4″ long inclined at 50°, is parallel to the vertical plane. Draw its projections.

3. An indefinite plane inclined at 40°, is perpendicular to the vertical plane. Represent it.

4. Two indefinite lines are at right angles; one is inclined at 50°, the other at 30°. Draw their plan.

5. Two planes are perpendicular to each other ; one is inclined at 40°, the other at 60°. Determine the vertical traces of these planes with reference to a ground line $x\ y$, making equal angles with their horizontal traces.

6. A regular pentagon of 2″ side lies in a plane inclined at 50°, and one diagonal $A\ C$ is inclined at 35°. Make an elevation on a plane parallel to the diagonal $A\ D$.

7. Three corners of a cube of 2·5″ edge are 0″, 1″, 1·5″ above the horizontal plane. Make an elevation on a plane parallel to a diagonal of the solid.

8. Each of three indefinite straight lines is perpendicular to the plane containing the other two; one is inclined at 50°, another at 30°. Draw the plan of the solid angle formed by them.

9. An irregular triangular pyramid has its six edges 1·6 in., 1·8 in., 2·7 in., 3 in., 3·2 in.; represent this solid when its longest edge is horizontal, and an adjacent face inclined at 30°.

10. A cylinder 4 in. long, its base 2·5 in. diameter, lies on its side on the horizontal plane; a cone 4 in. high, its base 2·5 in. diameter, stands on its base and touches the cylinder in a point equidistant from its ends; a sphere 1·5 in. radius also lies on the horizontal plane and touches the cone and the cylinder. Represent the combination in plan and elevation.

II.

1. Define accurately the terms Plan, Elevation, Section, and Trace.

2. How many regular solids are there? Name them, and give a reason for their number being limited.

3. Draw the plan and elevation of a point *A*, which is situated above the H. P., 2″ behind the V. P., and 3″ distant from *x y*.

4. A point *P* is 3·25″ above the H. P., 1·5″ in front of the V. P.; another, *Q*, is 2″ below the H. P., 3″ behind the V. P. The distance between the projectors, along *x y*, is 2·5″. Determine the true length, the inclination and the traces of the line *P Q*.

5. Show the plan and elevation of a straight line 3·5″ long, inclined at 35° to the H. P., 42° to the V. P.

6. The horizontal trace of a plane makes an angle of 30° with *x y*; draw the vertical trace, on the supposition that the plane is inclined at 65°. Determine the true angle between these traces.

7. A straight line is inclined 35°; determine a plane to contain it and inclined 50°.

8. The side of an equilateral triangle is 3″ and inclined at an angle of 25°; another side is inclined 45°. Draw its plan.

9. Complete the octohedron of which the triangle in Ex. 8 is a face.

10. A right pyramid, 5 in. high, has its axis inclined at 50° and one diameter, 4 in. long, of its hexagonal base, inclined at 25°. Draw its projections.

11. Determine the sphere circumscribing the pyramid in Ex. 10.

12. The edge of a cube is 3 in.; some of the faces of the solid are inclined at 45°, others at 70°. Draw its plan.

III.

1. The three plane angles of a trihedral angle (a spherical triangle) are 35°, 40°, 50°; determine its three dihedral angles (given the sides to find the angles by construction).

2. The three dihedral angles of a trihedral angle are 145°, 140,° and 130°; determine the plane angles (given the angles of a spherical triangle to find the sides by construction).

3. A circular table 3·5 feet in diameter and 3″ thick, is supported by four legs 3 inches square and 3 feet long, these legs are at the corners of a square of 2 feet side; show this table in Isometrical Projection. Scale $\frac{1}{12}$.

4. A reservoir is to be excavated on the side of a hill, tolerably flat on its surface for the extent required and sloping equably $\frac{1}{5}$ to the south; the top of the reservoir is to be a horizontal circle 30 yards in diameter in the clear, the centre being on the surface of the ground. The southern half of the reservoir to be formed by an embankment, 3 yards wide at the top, and 12 yards wide at the level of the bottom of the reservoir which is to be 6 yards deep. The northern half to be formed by cutting into the hill at a slope of $\frac{4}{3}$; show the work by a plan contoured for every two feet in level, adding a section of the work from N to S. Scale $\frac{1}{207}$.

5. A sphere of 1·5″ radius rests on the horizontal plane; determine its shadow cast by a light distant 8″ from the centre of the sphere, and 8″ above the horizontal plane.

6. Prove that if an ellipse be the isometrical projection of a circle, its minor axis, its isometrical diameter, and its major axis will be as $1 : \sqrt{2} : \sqrt{3}$.

7. Prove that if a line makes angles $\theta°$, $\phi°$, $\psi°$, with three rectangular axes, $\cos^2 \theta + \cos^2 \phi + \cos^2 \psi = 1$, and that the same expression holds if θ, ϕ, and ψ are the angles which a plane makes with three rectangular co-ordinate planes.

8. Prove that the orthogonal projections of two diameters of a circle which are at right angles will be conjugate diameters of the ellipse, which is the projection of that circle, and hence deduce the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, as the equation to the curve referred to conjugate diameters, a and b being the semi-conjugate diameters.

IV.

1. The plan $a\ b$, of the diameter of a sphere is 3 inches long; the point a is figured 25; b, 50. Draw the plan of the sphere, and determine a tangent plane at the point B.

2. From a point a, figured 40, draw the indefinite straight lines, AB and AC, at right angles to each other, AB inclined 30°, and AC 20°.

3. The horizontal trace of a plane makes an angle of 60°, and the vertical trace an angle of 35°, with $x\ y$; draw a straight line parallel to the plane, and inclined at 20°, which shall pass through a point 2″ high, and 2″ distant from the plane.

4. Through a point 3″ high and in the vertical plane of projection, draw a line making an angle of 30° with that plane and an angle of 45° with the horizontal.

5. From a point P, 20 units high, in a plane inclined at 50°, determine—

(1) A perpendicular to the plane.

(2) A straight line lying in the plane and having an inclination of 25°.

6. Determine the inclination of the plane containing the lines (1) and (2) Ex. 5 to the plane inclined at 50°.

7. Draw an equilateral triangle of 3″ side, and figure the corners 7, 16·5, 23; determine the plan of a point which is 3″ distant from each corner.

8. A pyramid 45 units high, with a square base of 5 units, stands on a plane inclined at 40° to the horizon, one edge of base of pyramid being inclined at 20°. Draw its plan.

9. Represent ground sloping 1 foot in 15 by contour lines parallel to the edge of the paper. Number the contours from the bottom of the paper, commencing with 10 and onwards. At the 12th contour draw a line making an angle of 60° with it and 35 feet long. This represents the trace of the crest of the parapet of a traverse, which is to be constructed of the following dimensions:—

Command of parapet 8′ 0″ over 12th contour.
" " 12′ 0″ at other end.
Thickness " 15′ 0″
Superior slope $\frac{1}{5}$, exterior slope $\frac{1}{1}$, side slopes 45°.
Banquette and other slopes as usual.

Scale $\frac{1}{180}$.

Construct and contour the traverse.

V.

1. Draw three lines meeting in a point and making oblique angles with one another. Figure these three lines so that they may be the plans of the edges of a solid right angle.

2. A tetrahedron, 4″ high, has for its base an equilateral triangle ABC of 3″ edge. The face, VAB, is inclined at 65°, VBC at 75°. Draw a plan and elevation, and state the inclination of the other face.

3. Draw the plan of a pentagon of 2·5″ side from either of the following conditions:—

a. When two diagonals are inclined at 35° and 45°.

b. When one diagonal is horizontal, the other inclined at 50°.

N.B.—The diagonals referred to meet at one angle of the pentagon.

4. Draw the horizontal trace of a plane, making an angle of 35° with the ground line, the vertical trace making one of 50°. Find the intersection with this plane of a line parallel to xy, 2″ above the H. P., and 1·5″ distant from the V. P.

5. Draw the plan of a line 4″ long, inclined at $\frac{1}{5}$, and two planes containing it, sloping in opposite directions, and inclined, the one at $\frac{1}{3}$, the other at $\frac{3}{2}$.

6. Find the angle between the two planes in Ex. 5.

7 A square pyramid, the height of which is 4″, and the edge of the base 2″, has its axis inclined at 60°, and one of the diagonals of the base at 15°. Its top is cut off by a plane inclined at 50° passing through the axis, 2″ from the vertex. Draw the plan of the frustum.

8. Draw the isometrical projection of a doorway 7 ft. high, 3 ft. wide, in a wall 18 in. thick, having a semicircular top, and approached by three steps, each 6 in. high and 1 ft. deep.

9. One of the faces of a regular octohedron, of 2 inches edge, is vertical, and an edge of that face inclined at 20°; draw a plan of the solid, and show upon it each even-numbered contour, making the lowest point zero.

10. From each extremity of a straight line A B, 2″ long, arcs A C, B C are described, with radii equal to A B. Draw the plan of the figure A C B when it lies in a plane inclined at 40°, A B being inclined at 25°.

11. Project the shadow thrown by the fig. A C B in Ex. 10 on a horizontal plane passing through its lowest angle, the light being supposed to proceed from a point 3″ vertically above the same angle.

Cooper's Hill, 1875 and 1876.

1. The horizontal traces of two planes, having inclinations of 35° and 55° respectively, intersect at an angle of 60°. Determine the intersection of the planes, its inclination to the horizon, and the angle contained between the planes.

2. A tetrahedron (*i.e.* a solid bounded by 4 equal equilateral

Fig. 97.

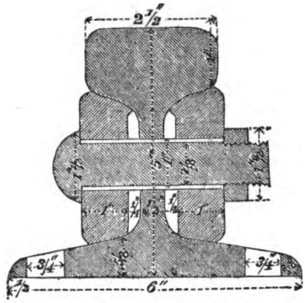
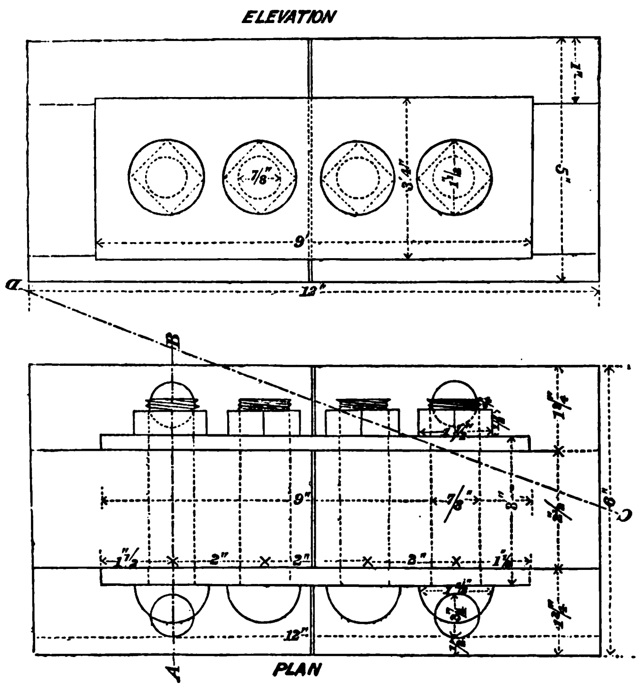


Fig. 98.



triangles) of 3 inches edge, has one of its edges horizontal and an adjacent edge inclined at 40°. Draw a plan of the tetrahedron.

3. Explain the principle upon which isometrical projection is based, and make an isometrical projection of two rails connected by fish plates. Draw also a sectional elevation on the given line CD. (*See Figures* 97, 98.)

4. Explain the method of making a scale for an isometrical drawing, and prove that the method is accurate.

5. Draw the isometrical projection of a square truncated pyramid, the base of which has a side of 4 inches, the top a side of 2 inches, the height being 3 inches.

6. Draw an isometrical projection of the box shown in Figs. 99 and 100. The lid is thrown back, as shown in section, and the

Fig. 99.

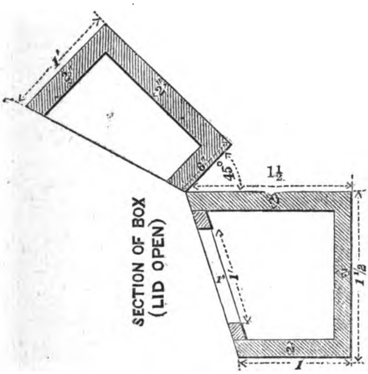
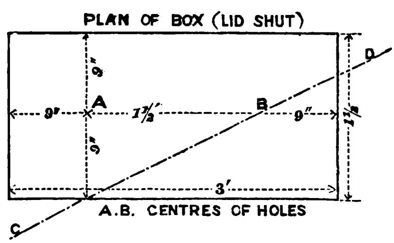


Fig. 100.



top of the lower portion of the box is covered with a piece of wood 1 inch thick, in which there are two circular holes of 1 foot in diameter.

Scale $\frac{1}{8}$ full size.

7. Draw a sectional elevation of the box, with the lid open, on the line CD.

Scale $\frac{1}{10}$ full size.

N.B.—*The drawings in 6 and 7 may be shaded with Indian ink, for which, if well done, extra marks will be allowed.*

8. Two lines 3 and 4 inches long and inclined to the horizon at 25° and 55°, bisect each other at right angles. Draw the plan of the lines, and the traces or scale of slope of the plane containing them. Determine the inclination of this plane to the horizon.

9. Draw the plan of a cube of 3 inches edge, when one face is inclined at 40° to the horizon and an edge in that face at 20°.

1877.

1. The plane of an equilateral triangle, 3 inches side, is inclined at 35°; one of its sides at 20°. Draw the plan.

2. Draw a line parallel to the bottom edge of your paper. This represents the intersection of a vertical and a horizontal plane. From any point in this line draw two lines, one on each side, making with it angles of 45° and 70°. These lines represent the vertical and horizontal traces of a plane. Determine its inclination to both planes.

3. Explain the nature of an isometric projection. In what respects does it differ from a perspective picture? For what purposes would you adopt it?

4. An ink-bottle of the form of a truncated square pyramid is made of $\frac{3}{8}$ inch plate glass; side of base 3 inches; length of sloping edge 2 inches; side of upper section $1\frac{3}{4}$ inch. The top is a square prism of the same glass, hinged at one of the far sides. It lies open to its full extent. Draw the isometric projection.

5. The plan of an equilateral triangle of 3 inches side is a triangle having two of its sides $2\frac{1}{2}$ and $1\frac{1}{2}$ inches, and the included angle 70°. Determine the scale of slope (or the traces) of the plane, in which the triangle lies and the inclination of this plane, and of the third side of the triangle, to the horizon.

6. Draw the plan of a right hexagonal prism 4 inches long, of which each side of the base measures $1\frac{1}{2}$ inches, supposing one side of the base to rest on the horizontal plane and a face containing that side to be inclined 50° to the horizon.

1879.

1. Draw the plan of a line 3 inches long inclined at 40° to the horizon, and draw a plane, containing the line and inclined at 65° to the horizon.

From one extremity of the plan of the line draw the plan of a perpendicular to the plane, the perpendicular being $2\frac{1}{2}$ inches long.

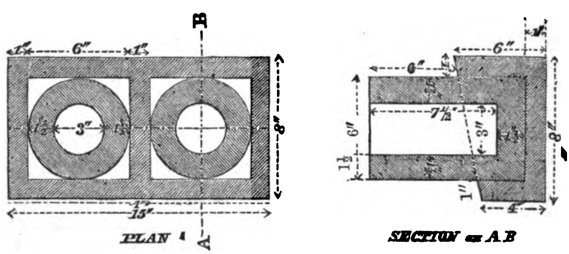
2. Draw the plan of a right triangular pyramid, the height being 4 inches, and the edges of the base 3 inches, the axis being inclined at 30° to the horizon, and one edge of the base being horizontal. Draw also a true form of the section of the pyramid made by a vertical plane passing through the centre of the axis, and perpendicular to its plan.

3. Two planes inclined to the horizon at 40° and 60° respectively have a common intersection inclined at 25° to the horizon. Determine the true angle contained by the planes.

From the same point in the common intersection draw a line in each plane inclined at 30° to the horizon, and determine the true angle contained by these lines.

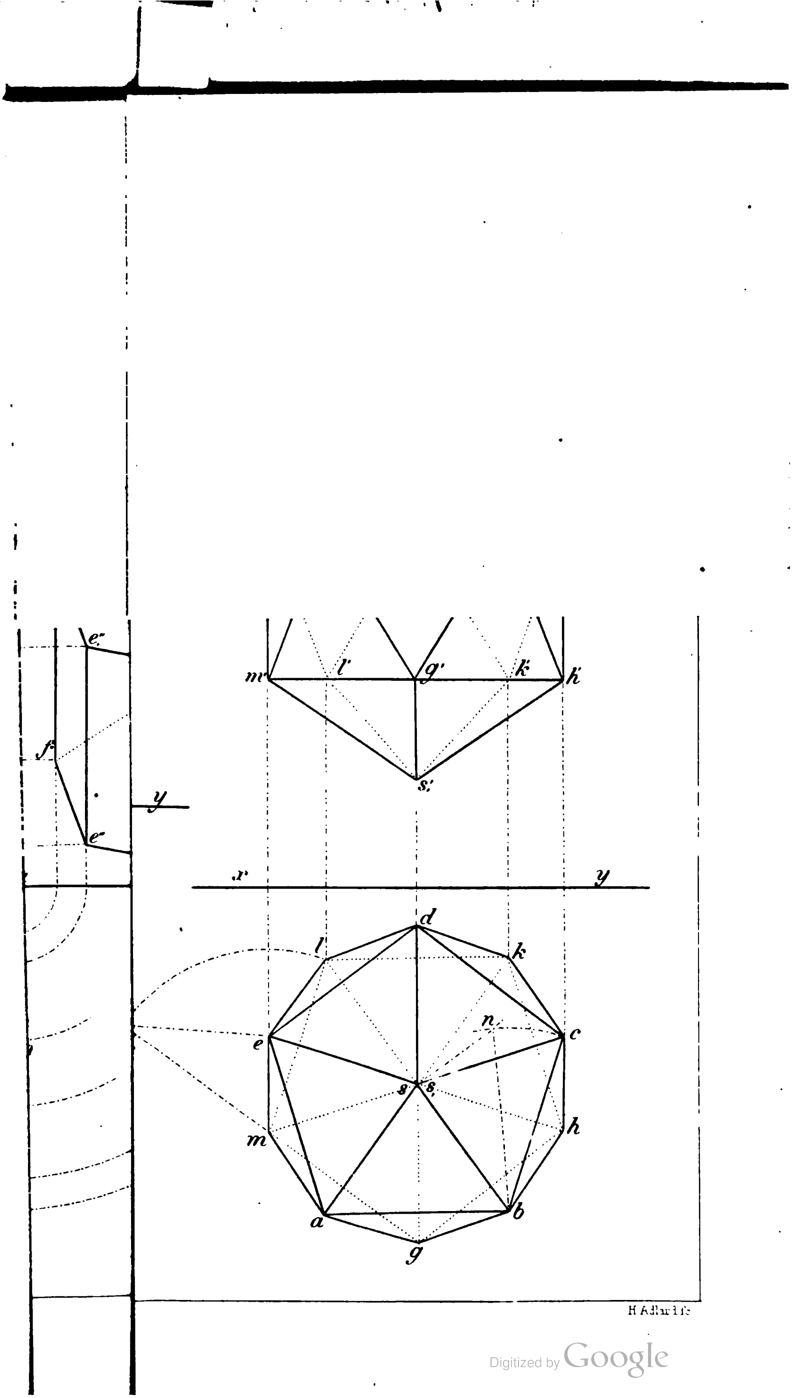
4. Make an isometric projection of the tray and jars shown in the accompanying diagrams.

Fig. 101.



True scale $\frac{1}{3}$.

Spottiswoode & Co., Printers, New-street Square, London.



H Adam fc.

Fig. 4